Research Training Group – Uncertainty and Randomness in Algorithms, Verification, and Logic
UnRAVeL

Chair for Logic and Theory of Discrete Systems

RWTH AACHEN UNIVERSITY

# Homomorphism Tensors and Linear Equations


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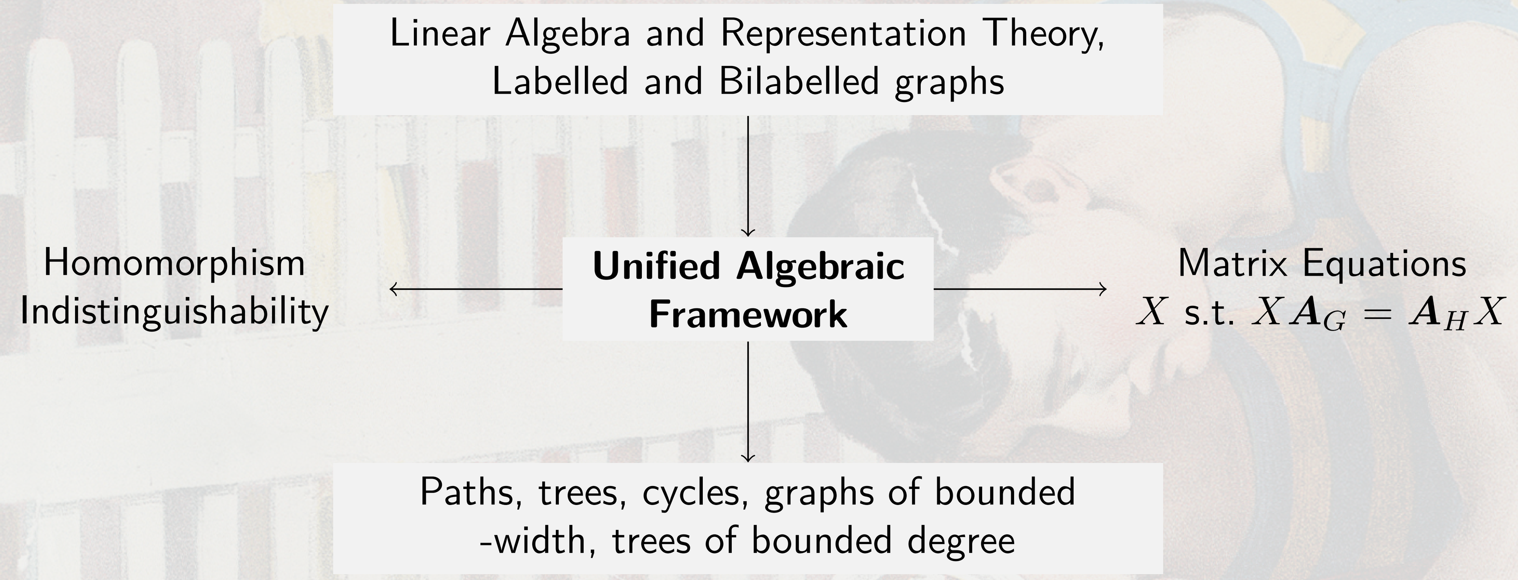
## Motivation

Lovász (1967) proved that graphs $G$ and $H$ are isomorphic if and only if they are *homomorphism indistinguishable* over all graphs $F$ the number of homomorphisms $F \to G$ equals the number of homomorphisms $F \to H$.

Homomorphism indistinguishability over restricted graph classes gives rise to a wide range of equivalence relations which can be characterised in terms of systems of equations. For example, graphs $G$ and $H$ are homomorphism indistinguishable over cycles/trees/path if and only if the system $X\boldsymbol{A}_G = \boldsymbol{A}_H X$ has an invertible/doubly-stochastic/pseudo-stochastic solution $X \in \mathbb{C}^{V(H)\times V(G)}$. We set out to provide a uniform explanation for such results.




## Labelled Graphs and Homomorphism Tensors

A *labelled graph* $\boldsymbol{F}$ is a tuple of a graph $F$ and a vertex $u \in V(F)$. Given a graph $G$, the *homomorphism tensor* of $\boldsymbol{F}$ is $\boldsymbol{F}_G \in \mathbb{C}^{V(G)}$ where

$$\boldsymbol{F}_G(v) \coloneqq \text{number of homomorphisms } h\colon F \to G \text{ such that } h(u) = v$$

for all $v \in V(G)$. This can be extended to *bilabelled graphs* $\boldsymbol{F} = (F, u_1, u_2)$ which carry an *in-label* $u_1 \in V(F)$ and an *out-label* $u_2 \in V(F)$. Their homomorphism tensors $\boldsymbol{F}_G$ represent matrices in $\mathbb{C}^{V(G)\times V(G)}$.

**Example** For every graph $G$, the homomorphism tensor $\boldsymbol{A}_G$ of the bilabelled graph $\boldsymbol{A} = \overset{1}{\bullet}\!-\!\overset{2}{\bullet}$ is the adjacency matrix of $G$.

## Operations

Combinatorial operations on (bi)labelled graphs correspond to algebraic operations on homomorphism tensors.

- The *sum-of-entries* $\operatorname{soe} \boldsymbol{F}_G$ equals $\hom(F, G)$, the homomorphism count of the *underlying unlabelled* graph $F$ of $\boldsymbol{F}$.
- The *matrix product* $\boldsymbol{F}_G \cdot \boldsymbol{F}'_G$ equals the homomorphism matrix of the bilabelled graph obtained from $\boldsymbol{F}$ and $\boldsymbol{F}'$ by *series composition*.
- The *Schur product* $\boldsymbol{F}_G \odot \boldsymbol{F}'_G$ equals the homomorphism vector of the labelled graph obtained from $\boldsymbol{F}$ and $\boldsymbol{F}'$ by *gluing*.

**Example** The bilabelled graph $\overset{1}{\bullet}\!-\!\bullet\!-\!\overset{2}{\bullet}$ results from the series composition $\overset{1}{\bullet}\!-\!\overset{2}{\bullet} \cdot \overset{1}{\bullet}\!-\!\overset{2}{\bullet}$. Its homomorphism matrix is $\boldsymbol{A}_G^2 = \boldsymbol{A}_G \cdot \boldsymbol{A}_G$.

## Inner-Product Compatible Graph Classes

Using linear algebra, we obtain matrix equations for homomorphism indistinguishability over classes of labelled graphs $\mathcal{R}$ which are

- *inner-product compatible*, i.e. for all $\boldsymbol{R}, \boldsymbol{S} \in \mathcal{R}$ the homomorphism counts from the graph obtained by gluing $\boldsymbol{R}$ and $\boldsymbol{S}$ and forgetting labels, are equal to the homomorphism counts from some graph in $\mathcal{R}$, and
- *$\boldsymbol{A}$-invariant*, i.e. for every labelled graph $\boldsymbol{R} = (R, u) \in \mathcal{R}$, the labelled graph $\boldsymbol{A} \cdot \boldsymbol{R}$ obtained by adding a fresh vertex $u'$ to $R$, adding the edge $uu'$, and placing the label on $u'$, is also in $\mathcal{R}$.

**Example** The family of labelled paths with labels at end vertices is inner-product compatible. For example,

$$\operatorname{soe}\left(\overset{1}{\bullet}\!-\!\bullet \odot \overset{1}{\bullet}\!-\!\bullet\!-\!\bullet\right) = \operatorname{soe}\left(\bullet\!-\!\overset{1}{\bullet}\!-\!\bullet\!-\!\bullet\right) = \bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet = \operatorname{soe}\left(\overset{1}{\bullet}\!-\!\bullet\!-\!\bullet\!-\!\bullet\right)$$

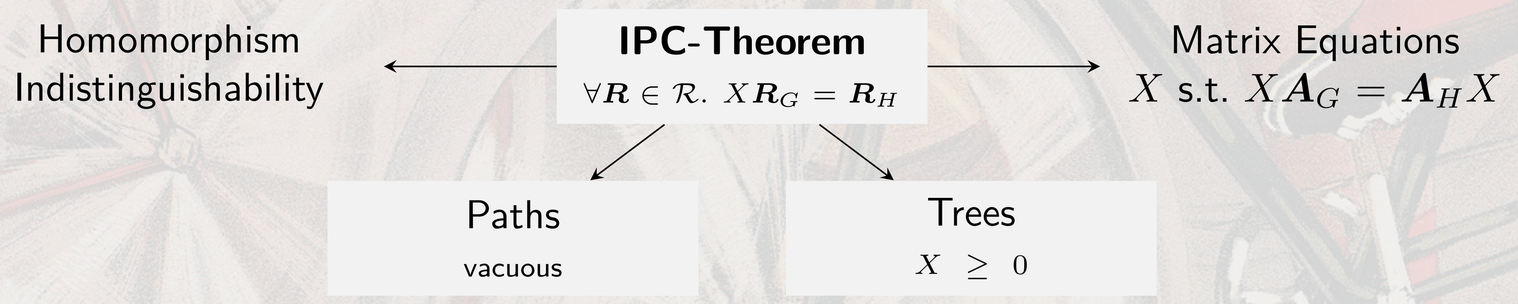
It is also $\boldsymbol{A}$-invariant. For example, $\boldsymbol{A} \cdot \overset{1}{\bullet}\!-\!\bullet = \overset{1}{\bullet}\!-\!\overset{2}{\bullet} \cdot \overset{1}{\bullet}\!-\!\bullet = \overset{1}{\bullet}\!-\!\bullet\!-\!\bullet$.

**IPC-Theorem** Let $\mathcal{R}$ be an inner-product compatible and $\boldsymbol{A}$-invariant family of labelled graphs containing $\overset{1}{\bullet}$. Then for graphs $G$ and $H$ the following are equivalent:

1. $G$ and $H$ are homomorphism indistinguishable over $\mathcal{R}$,
2. There exists a pseudo-stochastic $X \in \mathbb{Q}^{V(H)\times V(G)}$ such that $X\boldsymbol{R}_G = \boldsymbol{R}_H$ for all $\boldsymbol{R} \in \mathcal{R}$.

## Trees and Paths

We apply our theorem to the classes of trees and paths and prove known characterisation of homomorphism indistinguishability over these classes in a uniform manner. In particular, we find a combinatorial explanation for the obscurity that these characterisations differ only in the constraint $X \geq 0$.




## Trees of Bounded Degree

Characterising homomorphism indistinguishability over graph classes of bounded degree is a notoriously difficult problem. For trees of bounded degree, we prove the following.

**Theorem** For every $d \in \mathbb{N}$, there exist graphs $G$ and $H$ such that

- $G$ and $H$ are homomorphism indistinguishable over trees of degree $\leq d$ and
- $G$ and $H$ are not homomorphism indistinguishable over all trees.

In particular, it is not possible to simulate the 1-dimensional Weisfeiler–Leman algorithm (Colour Refinement) by counting homomorphisms from trees of bounded degree.

## Specht–Wiegmann Theorem

We use representation theory to derive novel matrix equations characterising homomorphism indistinguishability. The recipe is the following:

1. Definition of an involution monoid, for example the *path monoid* $\mathcal{P} = \{ \overset{1,2}{\bullet}, \overset{1}{\bullet}\!-\!\overset{2}{\bullet}, \overset{1}{\bullet}\!-\!\bullet\!-\!\overset{2}{\bullet}, \overset{1}{\bullet}\!-\!\bullet\!-\!\bullet\!-\!\overset{2}{\bullet}, \dots \}$.
2. For a graph $G$, define a representation $\mathcal{P} \to \mathbb{C}^{V(G)\times V(G)}$ mapping $\boldsymbol{P}$ to its homomorphism tensor $\boldsymbol{P}_G$.
3. The sum-of-entries of this representation counts the homomorphisms of interests. It can be interpreted as a character of a certain subrepresentation. The desired matrix equation arises from the following theorem:

**Theorem** Let $\varphi\colon \Gamma \to \mathbb{C}^{V\times V}$ and $\psi\colon \Gamma \to \mathbb{C}^{W\times W}$ be finite-dimensional representations of an involution monoid $\Gamma$. Then the following are equivalent:

1. For all $g \in \Gamma$, $\operatorname{soe}\psi(g) = \operatorname{soe}\varphi(g)$.
2. There exists a pseudo-stochastic $X \in \mathbb{C}^{W\times V}$ such that $X\varphi(g) = \psi(g)X$.

## Graphs of Bounded Pathwidth

Extending the known characterisation of homomorphism indistinguishability over graphs of treewidth $\leq k$ in terms of the existence of a *non-negative* solution to the Sherali–Adams-style relaxation $\mathsf{L}^{k+1}_{\mathsf{iso}}(G, H)$ of the ILP for graph isomorphism, we prove the following:

**Theorem** Let $k \in \mathbb{N}$. Graphs $G$ and $H$ are homomorphism indistinguishable over graphs of pathwidth $\leq k$ if and only if $\mathsf{L}^{k+1}_{\mathsf{iso}}(G, H)$ has a *rational* solution.

## Graphs of Bounded Treedepth

Our techniques yield a novel system of equations characterising homomorphism indistinguishable over graphs of bounded treedepth.

**Theorem** Let $k \in \mathbb{N}$. Graphs $G$ and $H$ are homomorphism indistinguishable over graphs of treedepth $\leq k$ if and only if the system of equations stated below has a rational solution.

$$\begin{aligned}
\sum_{v' \in V(G)} X(\boldsymbol{w}w, \boldsymbol{v}v') &= X(\boldsymbol{w}, \boldsymbol{v}) && \text{for all } w \in V(H) \text{ and } \boldsymbol{v} \in V(G)^\ell, \\
&&& \boldsymbol{w} \in V(H)^\ell \text{ where } 0 \leq \ell < k. \\
\sum_{w' \in V(H)} X(\boldsymbol{w}w', \boldsymbol{v}v) &= X(\boldsymbol{w}, \boldsymbol{v}) && \text{for all } v \in V(G) \text{ and } \boldsymbol{v} \in V(G)^\ell, \\
&&& \boldsymbol{w} \in V(H)^\ell \text{ where } 0 \leq \ell < k. \\
X(\boldsymbol{w}, \boldsymbol{v}) &= 0 && \text{if not } \boldsymbol{v}_i = \boldsymbol{v}_{i+1} \iff \boldsymbol{w}_i = \boldsymbol{w}_{i+1} \\
&&& \text{for all } i < k. \\
X((), ()) &= 1 && \text{for the empty tuple } ().
\end{aligned}$$

## References


[1] M. Grohe, G. Rattan and T. Seppelt. 'Homomorphism Tensors and Linear Equations'. In: *49th International Colloquium on Automata, Languages, and Programming, ICALP 2022, July 4-8, 2022, Paris, France.* Ed. by M. Bojańczyk, E. Merelli and D. P. Woodruff. Vol. 229. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. doi: 10.4230/LIPIcs.ICALP.2022.26.