Homomorphism Tensors and Linear Equations

A Cycle Race

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UnRAVeL Research Training Group – Uncertainty and Randomness in Algorithms, Verification, and Logic

RWTH AACHEN UNIVERSITY

DFG Deutsche Forschungsgemeinschaft

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1. Motivation


1. Motivation
2. Homomorphism Tensors
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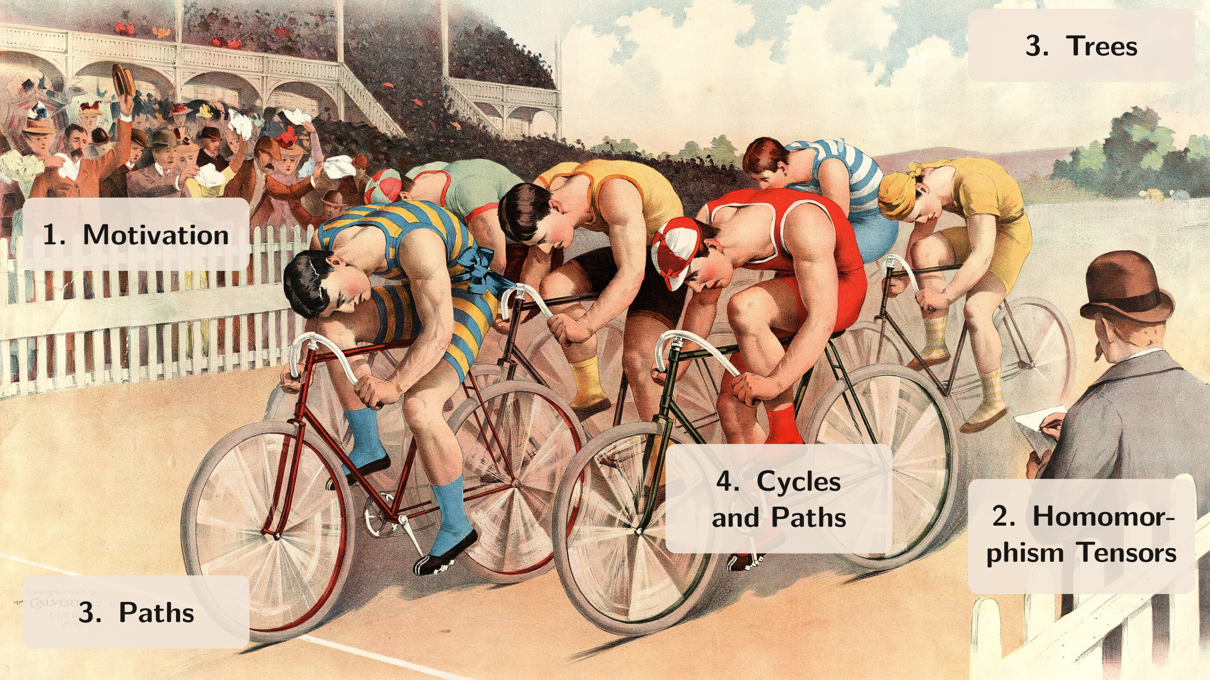
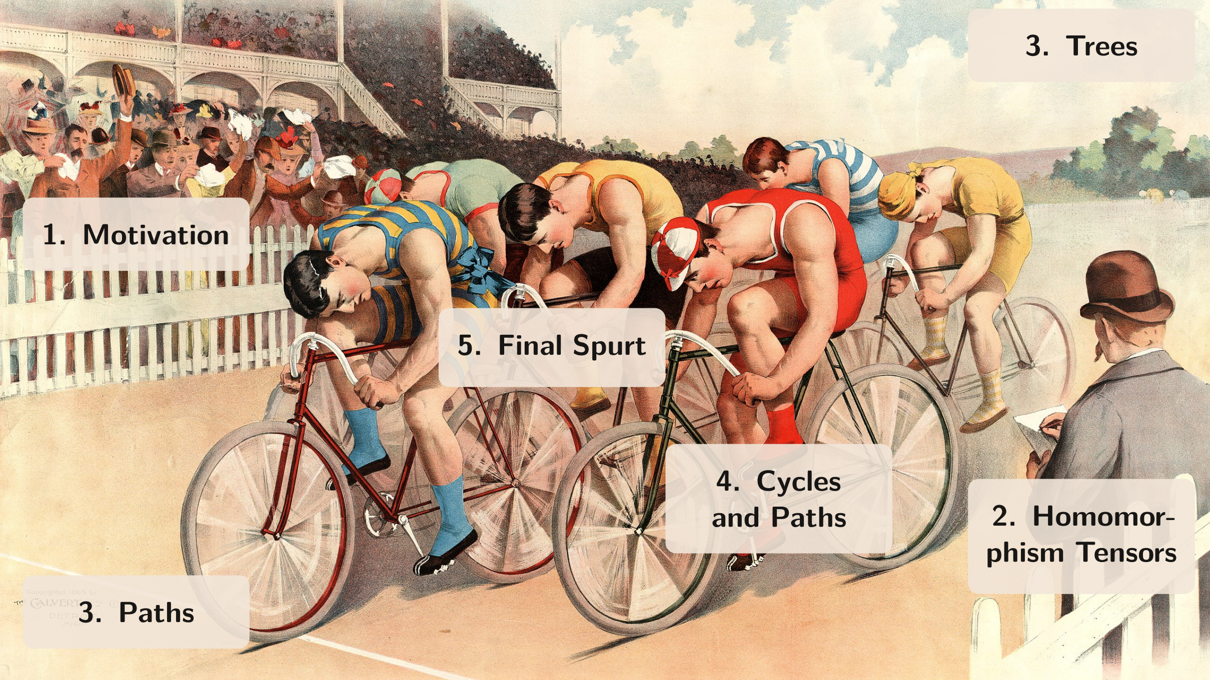
1. Motivation
2. Homomorphism Tensors
3. Paths
3. Trees


1. Motivation
2. Homomor-
phism Tensors
3. Trees
3. Paths
4. Cycles
and Paths


3. Trees
1. Motivation
5. Final Spurt
4. Cycles and Paths
2. Homomorphism Tensors
3. Paths

1. Motivation

Motivation: Numerical Graph Invariants

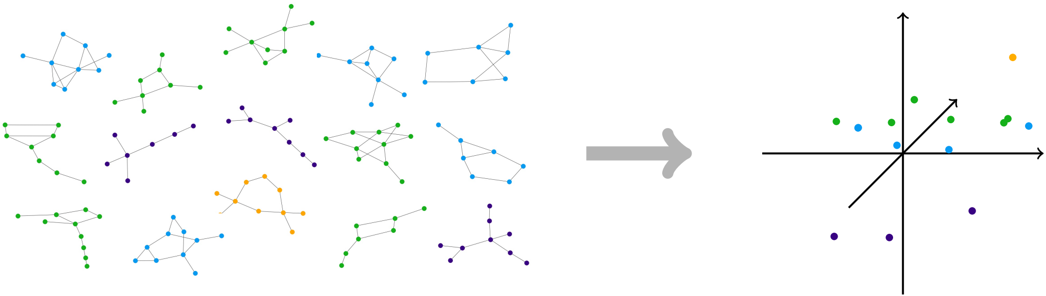
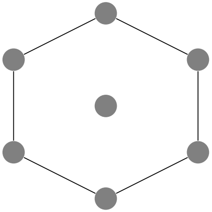


Figure: Homomorphism Embeddings
Figure from Grohe (2020)

Warming Up: Homomorphism Indistinguishability



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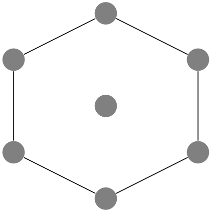

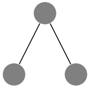

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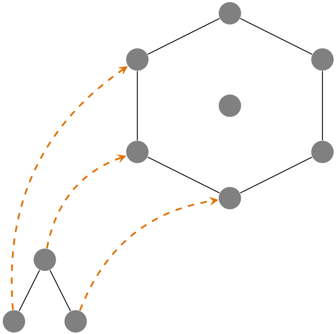

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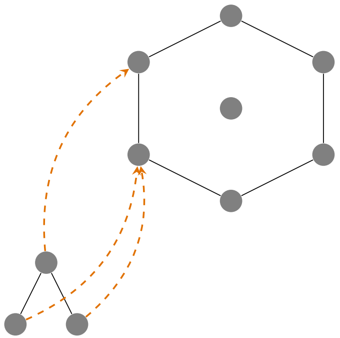

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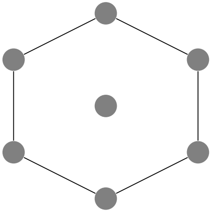

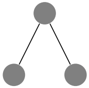

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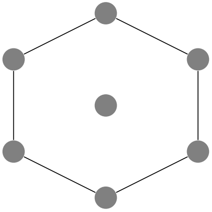
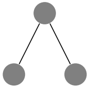
Warming Up: Homomorphism Indistinguishability





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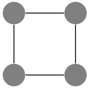

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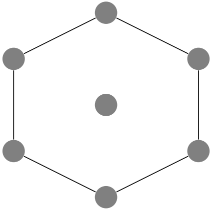
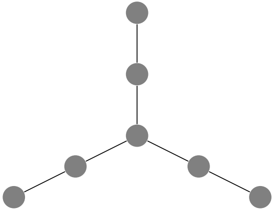
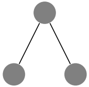
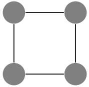
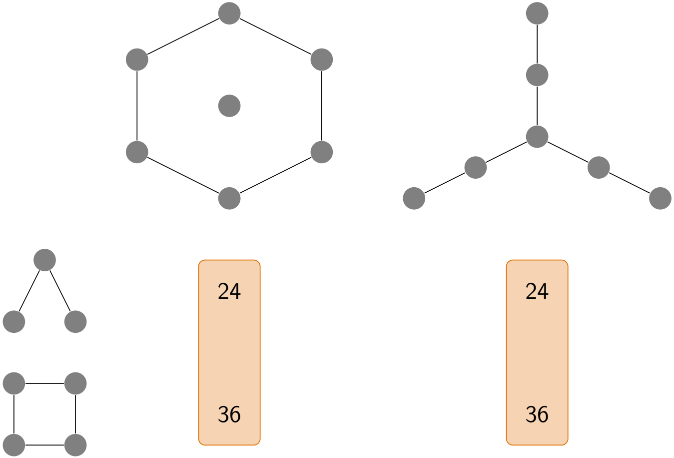
Warming Up: Homomorphism Indistinguishability

		
	24	24
	36	36

Warming Up: Homomorphism Indistinguishability



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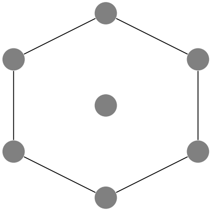

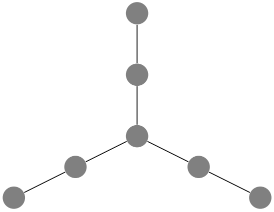

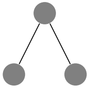

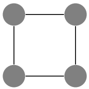

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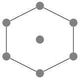
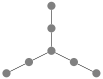
36	36

The graphs  and  are **homomorphism indistinguishable** over $\{$ ,  $\}$.

Motivation: Lovász

Homomorphism Indistinguishability

All Graphs ←— Lovász (1967) —→ Isomorphism

Motivation: Lovász

Homomorphism Indistinguishability		**Matrix Equations** X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$
All Graphs	Lovász (1967) $\longleftrightarrow$	X permutation matrix

Objective

Objective

Homomorphism Indistinguishability

Matrix Equations
X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$

Objective

Homomorphism Indistinguishability

Matrix Equations

X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$

Trees $\xleftrightarrow[\text{Dell et al. (2018)}]{\text{Dvořák (2010)}}$ X doubly-stochastic

$X \geq 0$, $X\mathbf{1} = \mathbf{1} = X^T\mathbf{1}$

Objective

Homomorphism Indistinguishability		**Matrix Equations** X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$
Trees	Dvořák (2010) ⟷ Dell et al. (2018)	X doubly-stochastic $X \geq 0$, $X\mathbf{1} = \mathbf{1} = X^T\mathbf{1}$
Paths	Dell et al. (2018) ⟷	X pseudo-stochastic $X\mathbf{1} = \mathbf{1} = X^T\mathbf{1}$

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Cycles	Folklore $\longleftrightarrow$	X orthogonal

Objective

Homomorphism Indistinguishability		Matrix Equations X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$
Trees	⟵ Instance-Specific Machinery ⟶	X doubly-stochastic $X \geq 0$, $X\mathbf{1} = \mathbf{1} = X^T\mathbf{1}$
Paths	⟵ Instance-Specific Machinery ⟶	X pseudo-stochastic $X\mathbf{1} = \mathbf{1} = X^T\mathbf{1}$
Cycles	⟵ Instance-Specific Machinery ⟶	X orthogonal

Objective

Unified Algebraic Framework

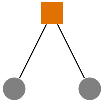
Homomorphism Indistinguishability

Matrix Equations
X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$

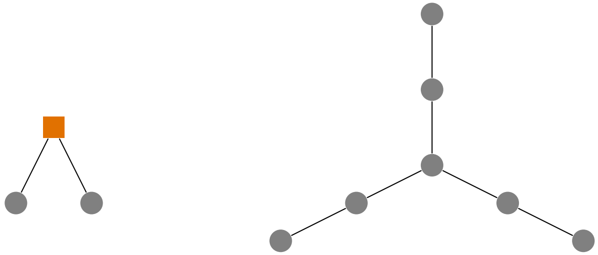
Trees	← Instance-Specific Machinery →	X doubly-stochastic $X \geq 0$, $X\mathbf{1} = \mathbf{1} = X^T\mathbf{1}$
Paths	← Instance-Specific Machinery →	X pseudo-stochastic $X\mathbf{1} = \mathbf{1} = X^T\mathbf{1}$
Cycles	← Instance-Specific Machinery →	X orthogonal

2. Homomorphism Tensors

Labelled Graphs and Homomorphism Tensors



Labelled Graphs and Homomorphism Tensors



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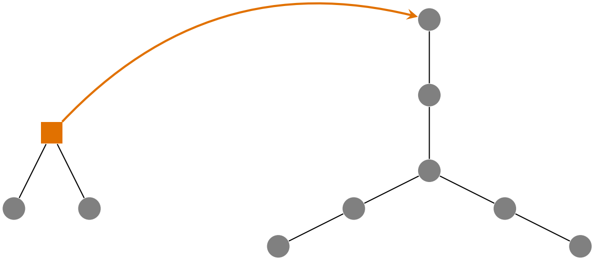

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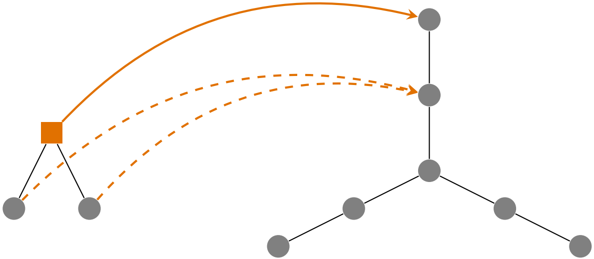

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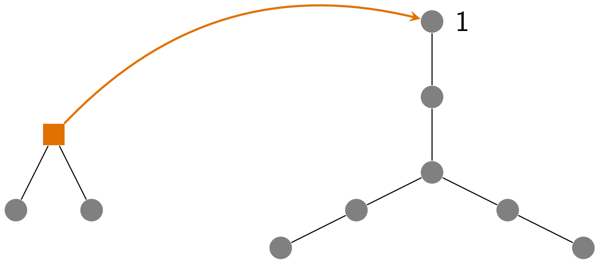

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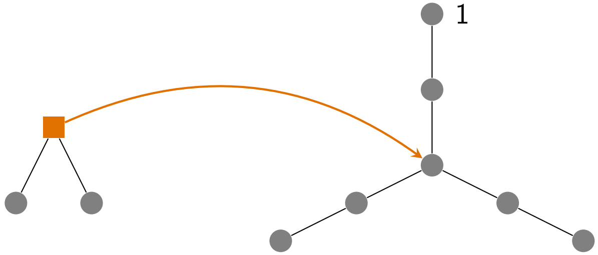

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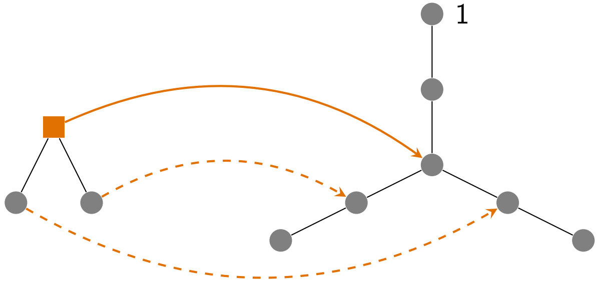

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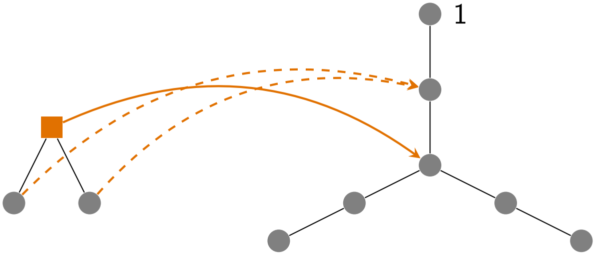

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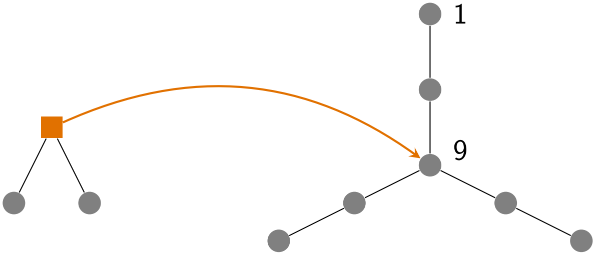
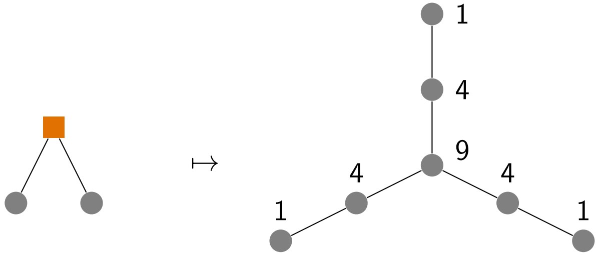
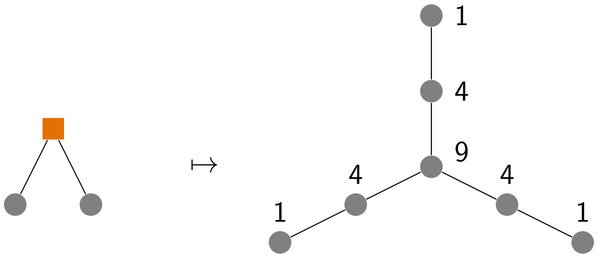


Labelled Graphs and Homomorphism Tensors

$$\mathcal{F} \longrightarrow \mathbb{C}^{V(G)}$$



Combinatorial and Algebraic Operations: Unlabelling and Sum-of-Entries



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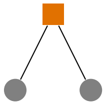
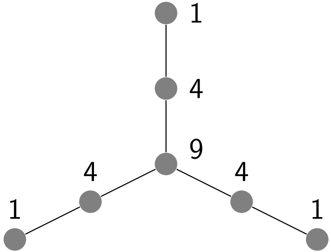
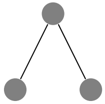


$\mapsto$



unlabel $\downarrow$



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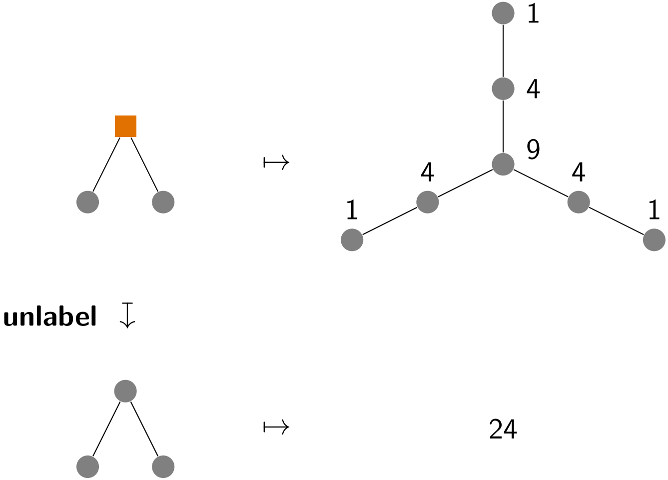

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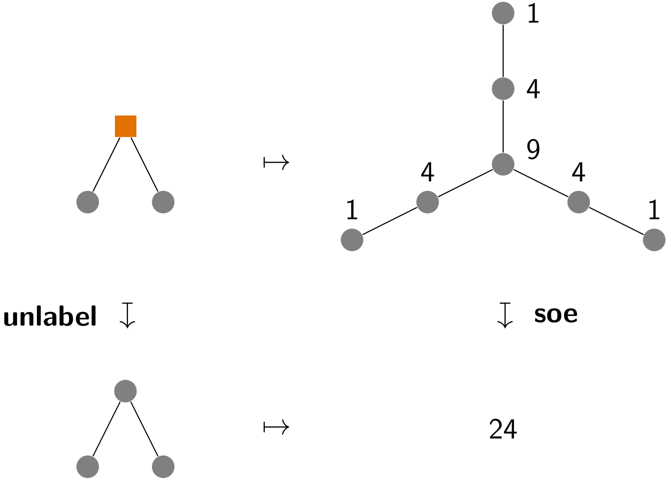
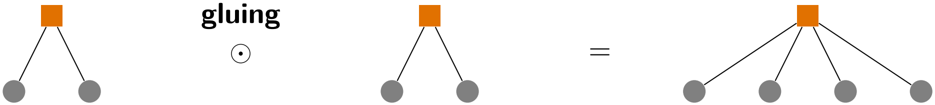


Combinatorial and Algebraic Operations: Gluing and Schur Product



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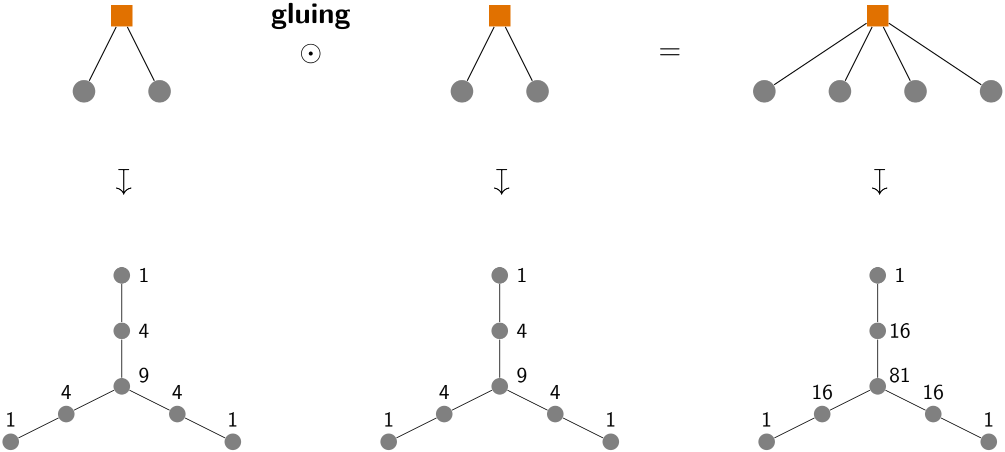

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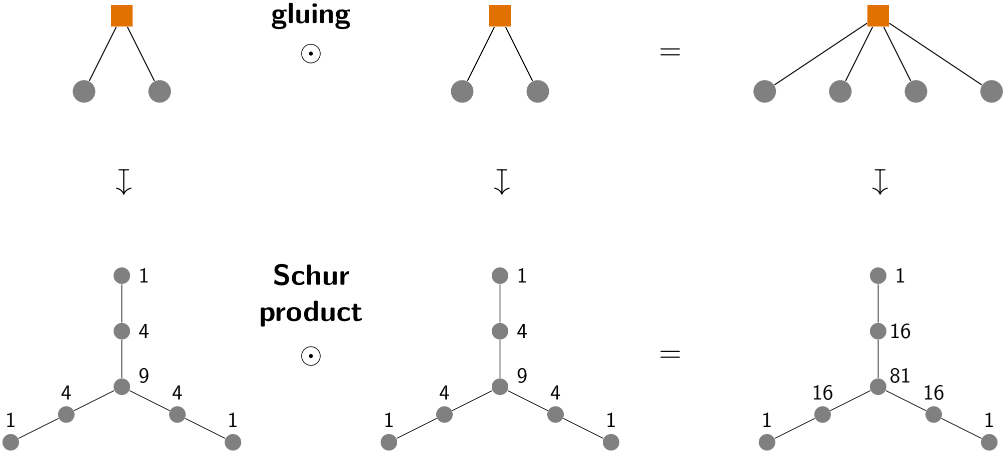

Homomorphism Tensors of Labelled and Bilabelled Graphs

Examples

- hom(■, G) $= \mathbf{1}_G \in \mathbb{C}^{V(G)}$, the all-ones vector.

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- hom(, G) $= \boldsymbol{A}_G \in \mathbb{C}^{V(G)\times V(G)}$, the adjacency matrix of G.

Remark

Homomorphism tensors resemble logical formulas: $\boldsymbol{A}_G(v_1, v_2) = 1 \iff G \models E(v_1, v_2)$.


3. Trees
3. Paths

First Theorem

Theorem

Let $\mathcal{R}$ be a set of labelled graphs.

1. *G and H are homomorphism indistinguishable over $\mathcal{R}$,*
2. *There exists a pseudo-stochastic X such that $X\boldsymbol{A}_G = \boldsymbol{A}_H X$ and $X\boldsymbol{R}_G = \boldsymbol{R}_H$ for all $\boldsymbol{R} \in \mathcal{R}$.*

First Theorem

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Let $\mathcal{R}$ be a set of labelled graphs that is inner-product compatible, $\boldsymbol{A}$-invariant, and contains ■.

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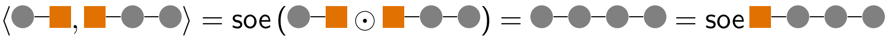
Definition (Inner-product compatible)

For all $\boldsymbol{R}, \boldsymbol{S} \in \mathcal{R}$ there exists $\boldsymbol{T} \in \mathcal{R}$ such that $\langle \boldsymbol{R}, \boldsymbol{S} \rangle = \operatorname{soe} \boldsymbol{T}$.



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Definition ($\boldsymbol{A}$-invariant)

For all $\boldsymbol{R} \in \mathcal{R}$ also $\boldsymbol{A} \cdot \boldsymbol{R} \in \mathcal{R}$.

Paths and Trees

The classes of paths and trees are inner-product compatible and $\boldsymbol{A}$-invariant.

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Homomorphism Indistinguishability ← **First Theorem** → Matrix Equations X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$

Paths and Trees

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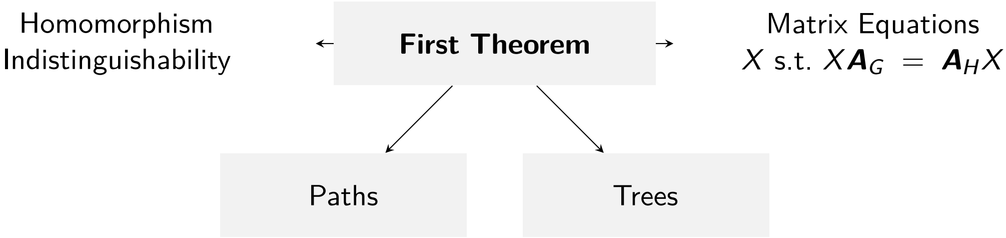

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Homomorphism Indistinguishability ← **First Theorem** $\forall \boldsymbol{R} \in \mathcal{R}.\ X\boldsymbol{R}_G = \boldsymbol{R}_H$ → Matrix Equations X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$

Paths

Trees

Paths and Trees

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Homomorphism Indistinguishability ← **First Theorem** $\forall \boldsymbol{R} \in \mathcal{R}.\ X\boldsymbol{R}_G = \boldsymbol{R}_H$ → Matrix Equations X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$

Paths
vacuous

Trees
$X \geq 0$

Bounded Degree Trees

Are bounded degree tree homomorphism counts as expressive as tree homomorphism counts?

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Theorem

For every $d \geq 1$, there exist graphs G and H such that

1. *G and H are homomorphism indistinguishable over* ***d-ary trees****, and*
2. *G and H are* ***not*** *homomorphism indistinguishable over* ***all trees****.*

Bounded Degree Trees

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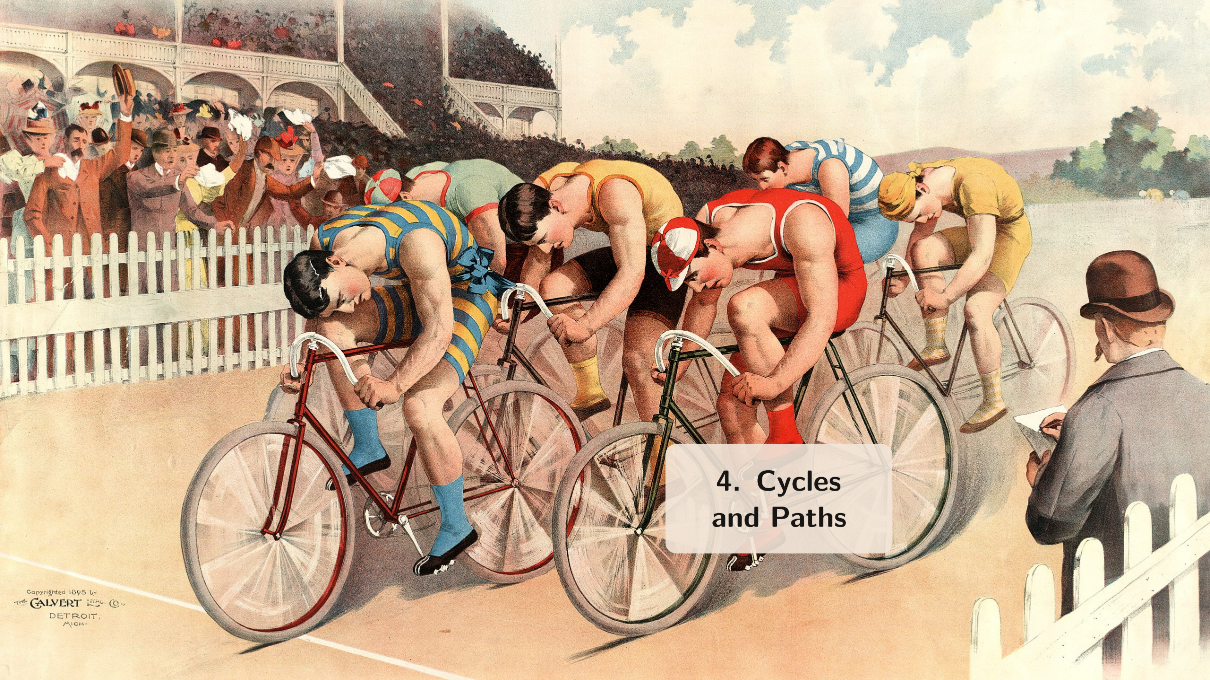
Theorem

For every $d \geq 1$, there exist graphs G and H such that

1. *G and H are homomorphism indistinguishable over **d-ary trees**, and*
2. *G and H are **not** homomorphism indistinguishable over **all trees**.*

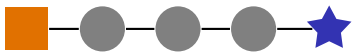
Corollary

It is not possible to simulate 1-WL (Colour Refinement) by counting homomorphisms from bounded degree trees.

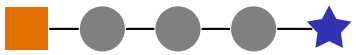


4. Cycles and Paths

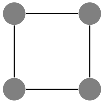
Combinatorial and Algebraic Operations: Glueing/Unlabelling and Trace



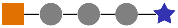
Combinatorial and Algebraic Operations: Glueing/Unlabelling and Trace



glue and unlabel ↧



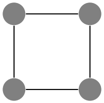
Combinatorial and Algebraic Operations: Glueing/Unlabelling and Trace



$\mapsto$

$$\begin{bmatrix} 12 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 6 & 0 & 5 & 0 & 5 & 0 \\ 4 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 5 & 0 & 6 & 0 & 5 & 0 \\ 4 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 5 & 0 & 5 & 0 & 6 & 0 \\ 4 & 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix}$$

glue and unlabel $\downarrow$



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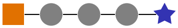

↦

12	0	4	0	4	0	4
0	6	0	5	0	5	0
4	0	2	0	1	0	1
0	5	0	6	0	5	0
4	0	1	0	2	0	1
0	5	0	5	0	6	0
4	0	1	0	1	0	2

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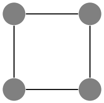

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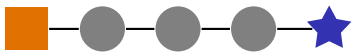

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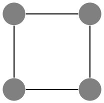
glue and unlabel ↧

↧ **trace**



↦

36

Crash Course on Representation Theory

- Let Γ be something like a group
- A **representation** of Γ is a homomorphism $\varphi\colon \Gamma \to \mathbb{C}^{n\times n}$

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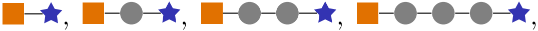
  - path monoid $\mathcal{P} = \{$  $, \ldots \}$
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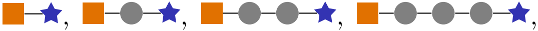
  - path monoid $\mathcal{P} = \{$  $, \ldots\}$
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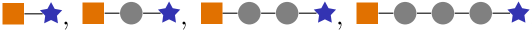
  - path monoid $\mathcal{P} = \{$  $, \ldots \}$
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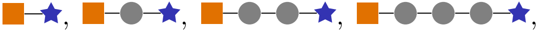
  - path monoid $\mathcal{P} = \{$  $, \dots \}$
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↕ Cycles

↕ X invertible s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$

Second Theorem: Paths (again)

Path homomorphism counts are tabulated by soe of the representation.

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Sum-of-Entries of φ and ψ are the same. ⟵ **Second Theorem** ⟶ X pseudo-stochastic s.t. $X\varphi = \psi X$

Second Theorem: Paths (again)

Path homomorphism counts are tabulated by soe of the representation.

Sum-of-Entries of φ and ψ are the same. ⟷ Paths

Second Theorem

X pseudo-stochastic s.t. $X\varphi = \psi X$ ⟷ X pseudo-stochastic s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$

Graphs of Bounded Pathwidth and Sherali–Adams Relaxation

Homomorphism Indistinguishability		**Matrix Equations** X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$
Trees	⟷ Dvořák (2010), Dell et al. (2018)	X doubly-stochastic $X \geq 0$, $X\mathbf{1} = \mathbf{1} = X^T\mathbf{1}$
Paths	⟷ Dell et al. (2018)	X pseudo-stochastic $X\mathbf{1} = \mathbf{1} = X^T\mathbf{1}$

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Treewidth $\leq k$	⟷ Atserias and Maneva (2012) / Grohe and Otto (2015)	$\mathsf{L}^{k+1}_{\text{iso}}(G, H)$ has non-negative solution
Pathwidth $\leq k$	⟵ Dell et al. (2018) ?	$\mathsf{L}^{k+1}_{\text{iso}}(G, H)$ has rational solution

Graphs of Bounded Pathwidth and Sherali–Adams Relaxation

Homomorphism Indistinguishability		**Matrix Equations** X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$
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Graphs of Bounded Pathwidth and Sherali–Adams Relaxation

Homomorphism Indistinguishability		**Matrix Equations** X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$
Trees	← **Unified Algebraic Framework** →	X doubly-stochastic $X \geq 0$, $X\mathbf{1} = \mathbf{1} = X^T\mathbf{1}$
Paths	← →	X pseudo-stochastic $X\mathbf{1} = \mathbf{1} = X^T\mathbf{1}$
Treewidth $\leq k$	← →	$\mathsf{L}_{\text{iso}}^{k+1}(G, H)$ has non-negative solution
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Graphs of Bounded Treedepth

Homomorphism Indistinguishability		**Matrix Equations** X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$
Treedepth $\leq d$	←— Grohe, Rattan, S. (2022) —→	Novel system of equations resembling ordered version of $\mathsf{L}^{k+1}_{\mathsf{iso}}(G, H)$

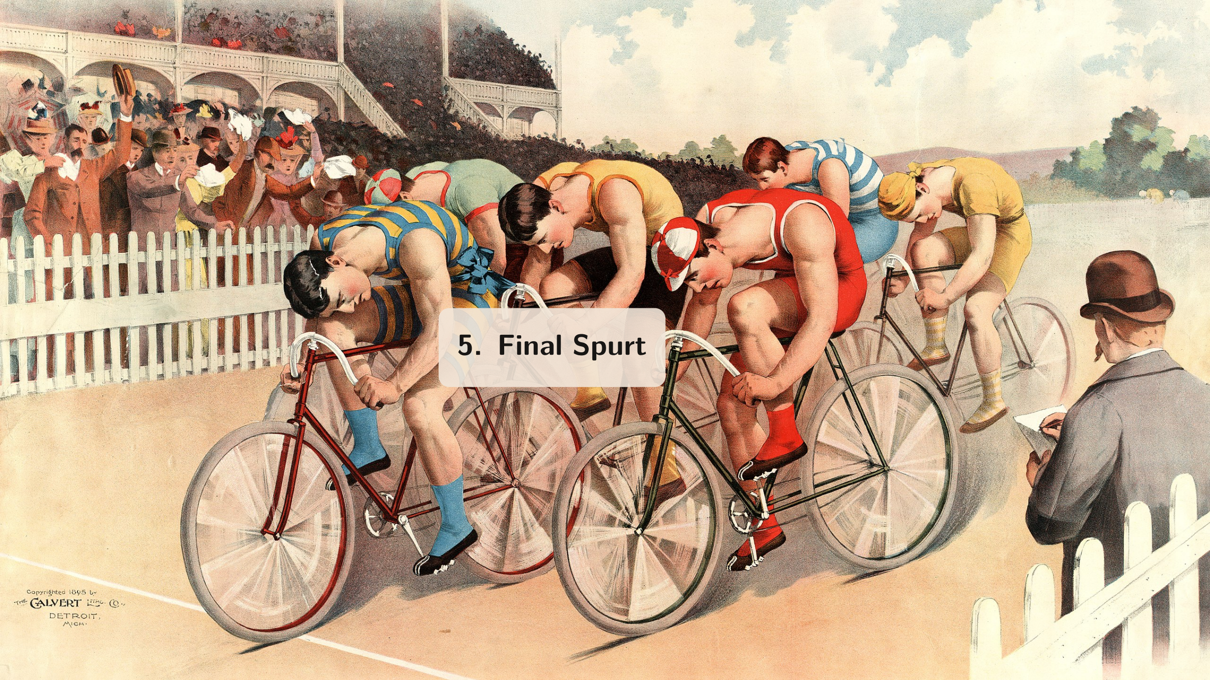
Graphs of Bounded Treedepth

Homomorphism Indistinguishability

Matrix Equations
X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$

Treedepth $\leq d$ ⟵ **Unified Algebraic Framework** ⟶ Novel system of equations resembling ordered version of $\mathsf{L}_{\text{iso}}^{k+1}(G, H)$

5. Final Spurt



Scoring

Homomorphism Indistinguishability ⟵ **Unified Algebraic Framework** ⟶ Matrix Equations X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$

Scoring

Linear Algebra and Representation Theory,
Labelled and Bilabelled graphs

↓

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Scoring

Linear Algebra and Representation Theory, Labelled and Bilabelled graphs

↓

Homomorphism Indistinguishability ← **Unified Algebraic Framework** → Matrix Equations X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$

↓

Paths, trees, cycles, graphs of bounded treewith and -depth, path-width, trees of bounded degree

Open Problems

- Extension to other graph classes
  - planar graphs, aligning with Mančinska and Roberson (2020)
  - relational structures and other graph classes via comonads of Dawar et al. (2021)
- Extension to graph similarity
  - metrics induced by homomorphism embeddings $G \mapsto (\hom(F, G) \mid F \in \mathcal{F})$ vs. matrix-based metrics $\min_X \|X\boldsymbol{A}_G - \boldsymbol{A}_H X\|$
  - Böker (2021) for trees

Thank you for your attention!

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