Homomorphism Indistinguishability for Comonadists

Comonads Online Meetup

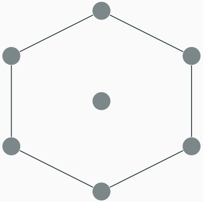
26 April 2023

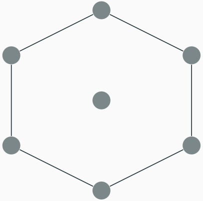
Tim Seppelt

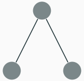
UnRAVeL Research Training Group – Uncertainty and Randomness in Algorithms, Verification, and Logic

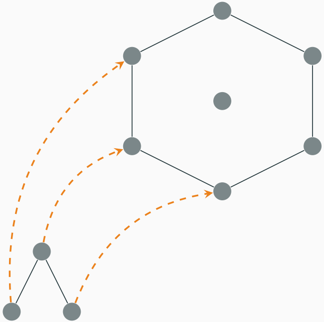
RWTH AACHEN UNIVERSITY

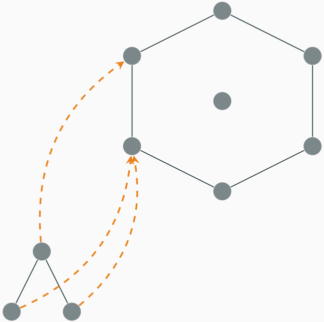
DFG Deutsche Forschungsgemeinschaft
German Research Foundation

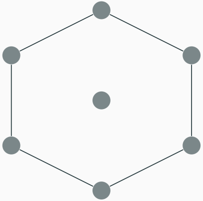


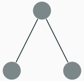












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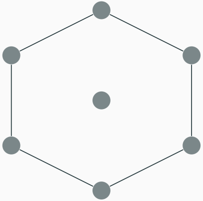

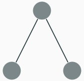

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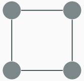

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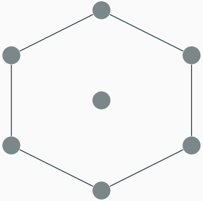
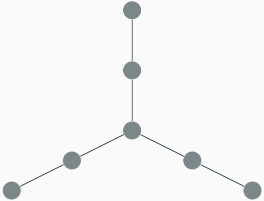
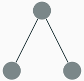
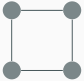
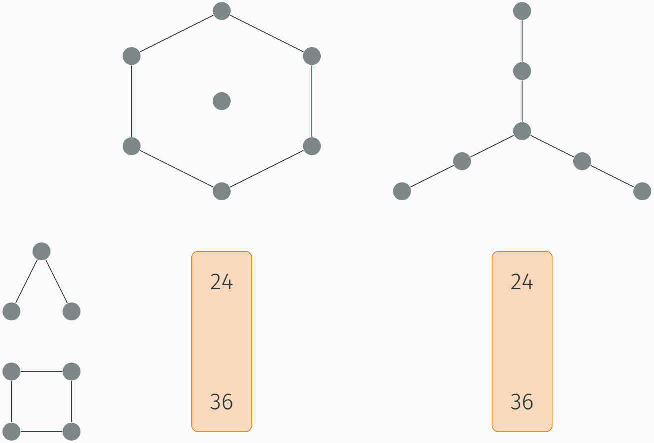
		
	24	24
	36	36



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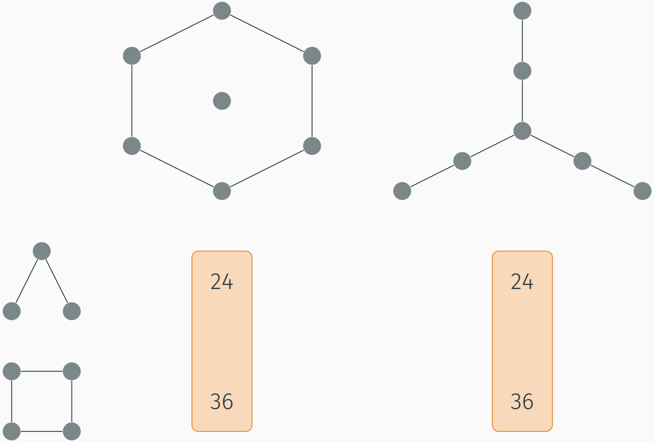

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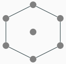
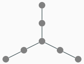
The graphs  and  are **homomorphism indistinguishable** over { , }.

Why Homomorphism Indistinguishability?

- Connections to graph properties in *finite model theory* and *algebraic graph theory*

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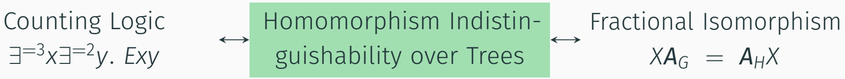

Counting Logic
$\exists^{=3}x\exists^{=2}y.\ Exy$

↔

Homomorphism Indistinguishability over Trees

↔

Fractional Isomorphism
$X\boldsymbol{A}_G = \boldsymbol{A}_H X$

Why Homomorphism Indistinguishability?

- Connections to graph properties in *finite model theory* and *algebraic graph theory*

Counting Logic
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↔
Homomorphism Indistinguishability over Trees
↔
Fractional Isomorphism
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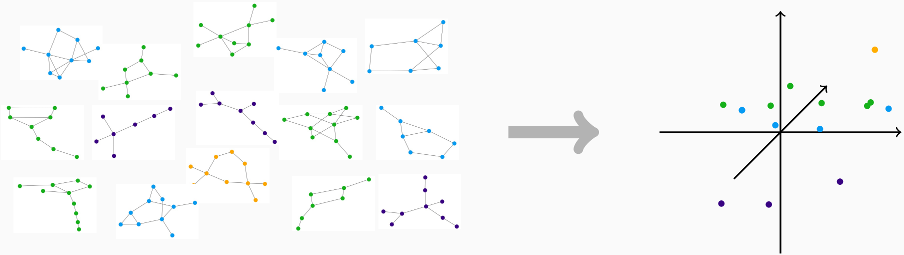
- Expressive *numerical graph invariants* for applications Illustration from Grohe (2020).



Outline

Matrix Equations for Homomorphism Indistinguishability

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Towards a Theory of Homomorphism Indistinguishability

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Towards a Theory of Homomorphism Indistinguishability

Open Questions

Matrix Equations for Homomorphism Indistinguishability

Matrix Equations for Homomorphism Indistinguishability

Homomorphism Indistinguishability		Matrix Equations X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$
All Graphs	$\xleftrightarrow{\text{Lovász (1967)}}$	X permutation matrix

Matrix Equations for Homomorphism Indistinguishability

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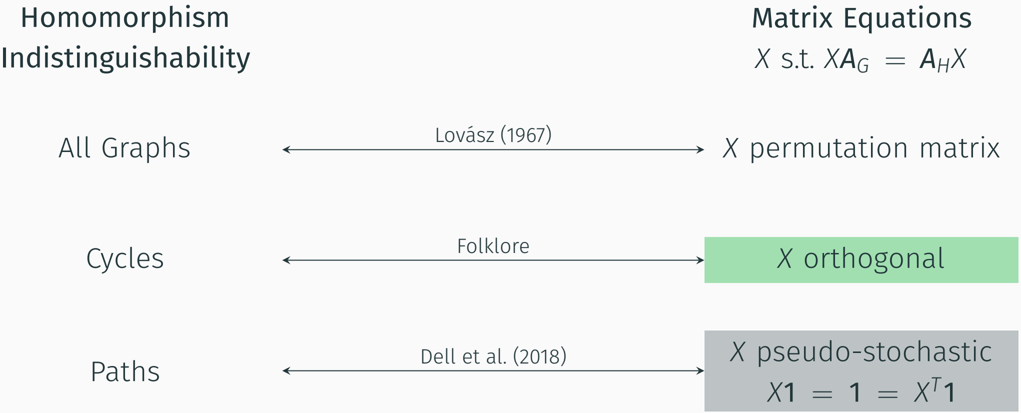
Cycles	$\xleftrightarrow{\text{Folklore}}$	X orthogonal

Matrix Equations for Homomorphism Indistinguishability



Homomorphism Indistinguishability

Matrix Equations

X s.t. $XA_G = A_H X$

All Graphs

Lovász (1967)

X permutation matrix

Cycles

Folklore

X orthogonal

Paths

Dell et al. (2018)

X pseudo-stochastic

$X\mathbf{1} = \mathbf{1} = X^T\mathbf{1}$

Matrix Equations for Homomorphism Indistinguishability

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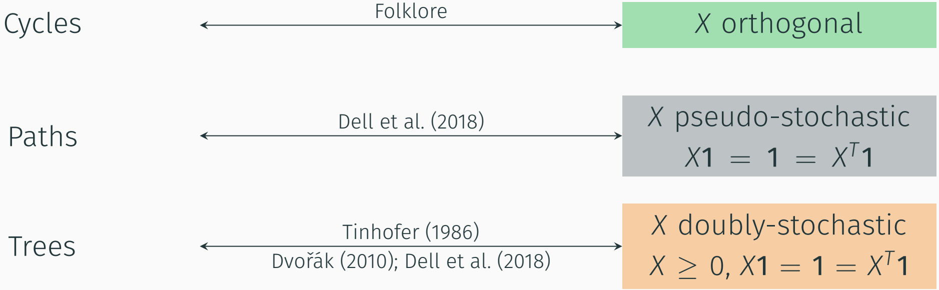
Trees	Tinhofer (1986) ⟷ Dvořák (2010); Dell et al. (2018)	X doubly-stochastic $X \geq 0$, $X\mathbf{1} = \mathbf{1} = X^T\mathbf{1}$

Matrix Equations for Homomorphism Indistinguishability

Homomorphism Indistinguishability

Matrix Equations

X s.t. $XA_G = A_H X$



Cycles ⟷ (Folklore) ⟷ X orthogonal

Paths ⟷ (Dell et al. (2018)) ⟷ X pseudo-stochastic $X\mathbf{1} = \mathbf{1} = X^T\mathbf{1}$

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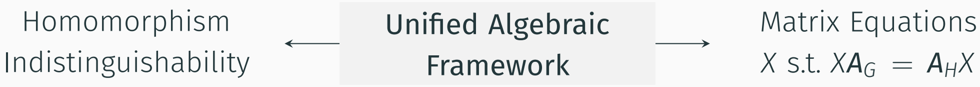
A Strategy for Matrix Equations Grohe, Rattan, S. (2022), Rattan & S. (2023)

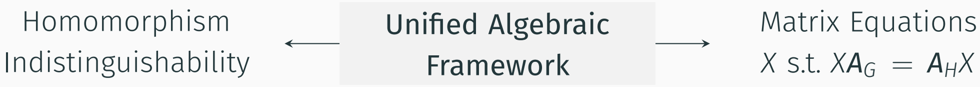


Homomorphism Indistinguishability

Unified Algebraic Framework

Matrix Equations X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$

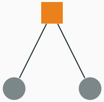
A Strategy for Matrix Equations Grohe, Rattan, S. (2022), Rattan & S. (2023)



Homomorphism Indistinguishability ⟵ **Unified Algebraic Framework** ⟶ Matrix Equations X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$

1. Construct family $\mathcal{F}$ of **(bi)labelled graphs**
2. Define suitable **operations**
3. Prove that $\mathcal{F}$ is **finitely generated** under operations
4. Define **representation** and recover system of equations

Labelled Graphs and Homomorphism Vectors



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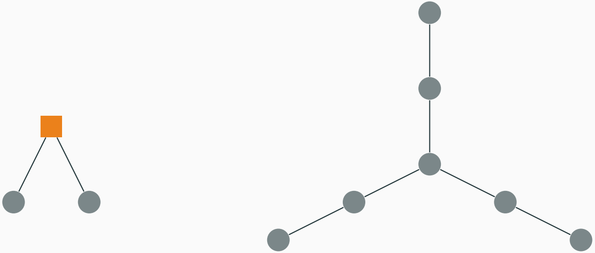

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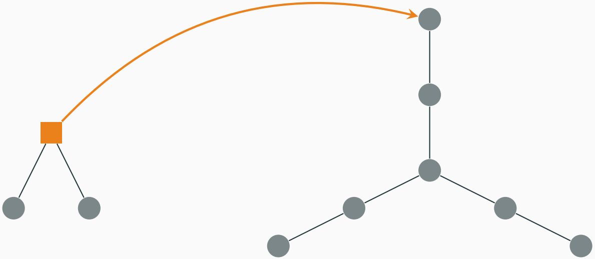

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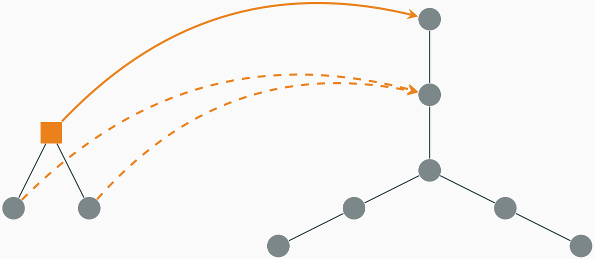

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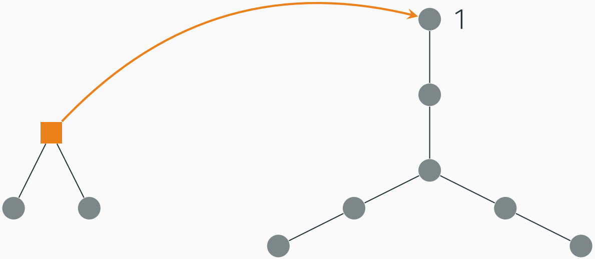

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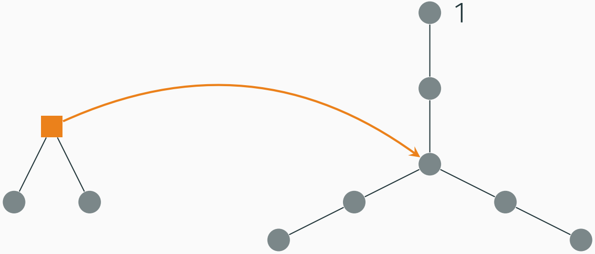

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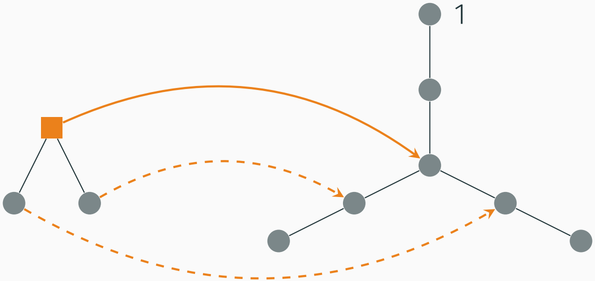

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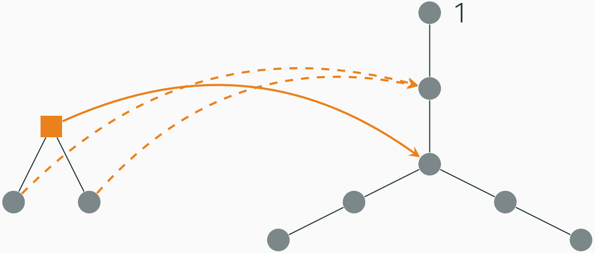

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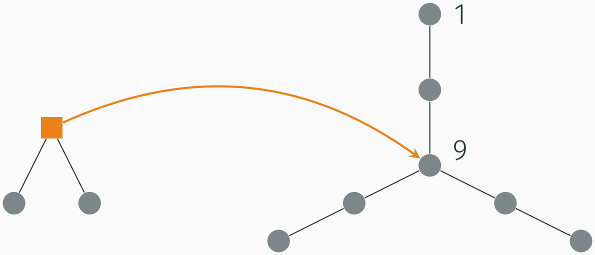
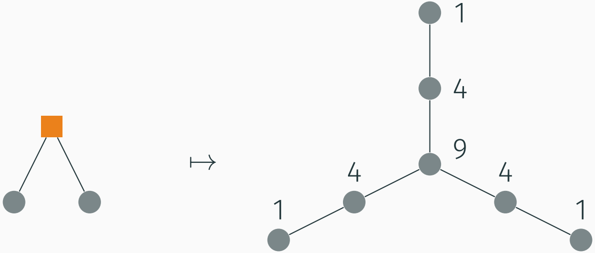
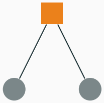


Labelled Graphs and Homomorphism Vectors

$$\mathcal{F} \longrightarrow \mathbb{C}^{V(G)}$$



Combinatorial and Algebraic Operations: Unlabelling and Sum-of-Entries



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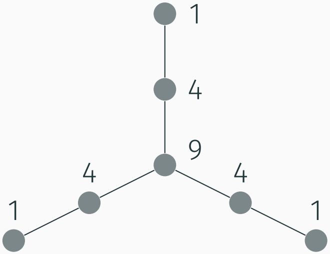

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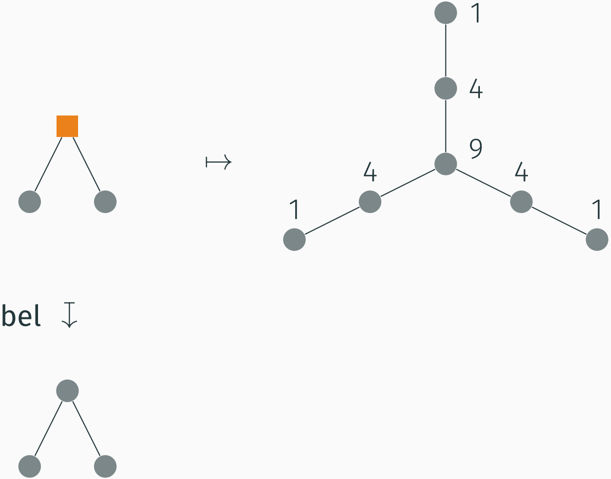
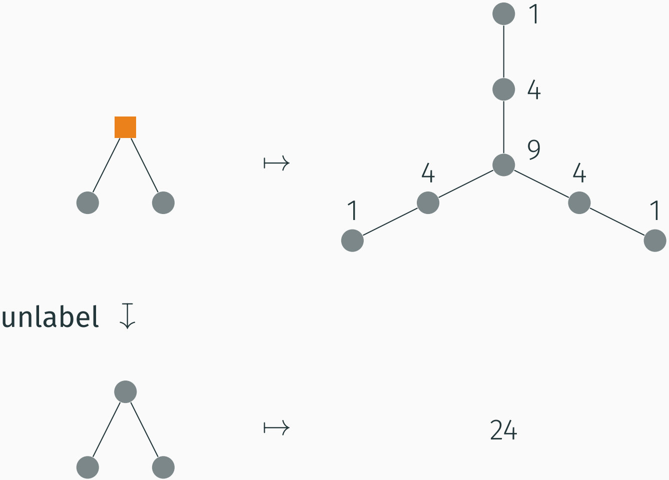


unlabel ↧

Combinatorial and Algebraic Operations: Unlabelling and Sum-of-Entries



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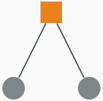

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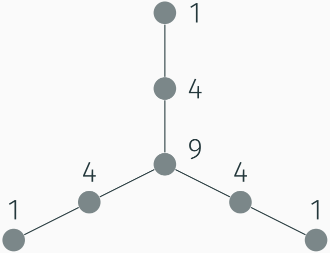
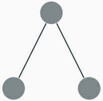


unlabel ↧

↧ sum of entries



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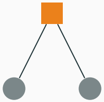
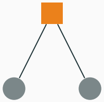
Combinatorial and Algebraic Operations: Gluing and Schur Product



gluing

$\odot$



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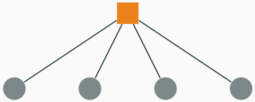

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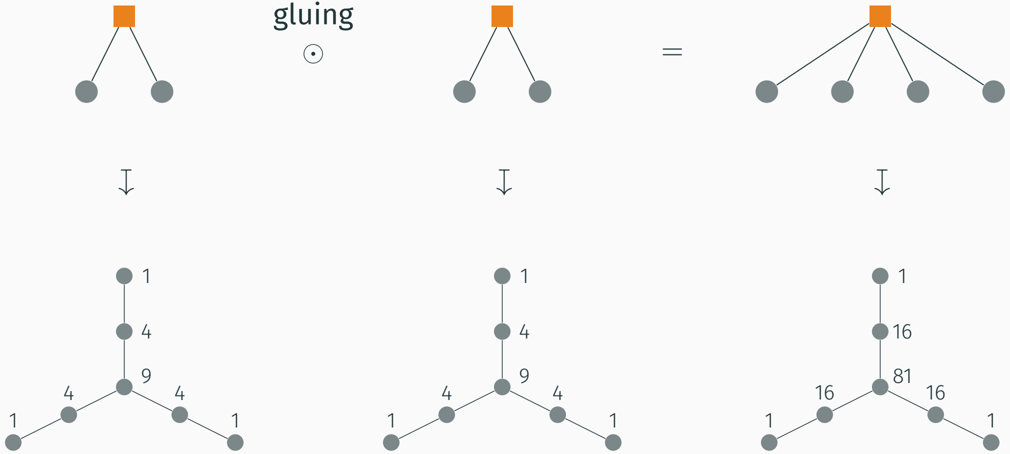

gluing

$\odot$ $=$

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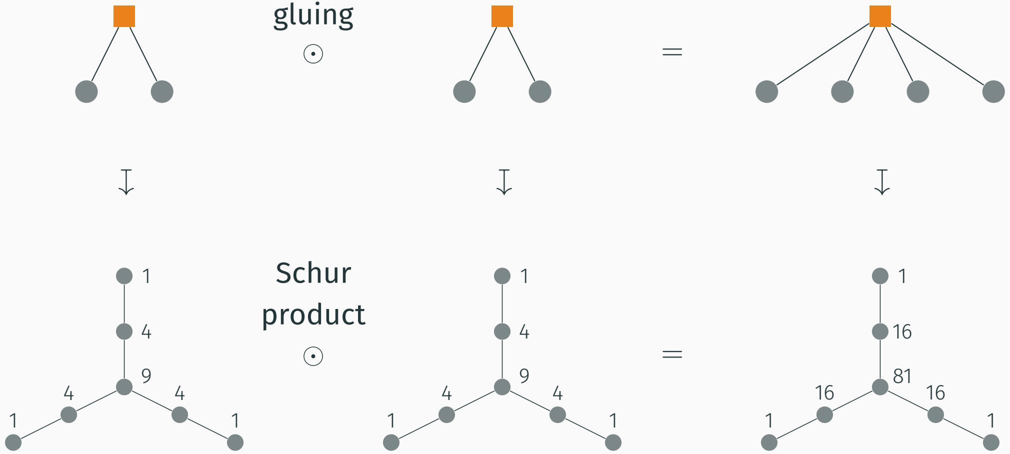
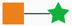


gluing $\odot$ $=$

$\downarrow$

Schur product $\odot$ $=$

Bilabelled Graphs and Homomorphism Matrices



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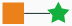
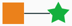
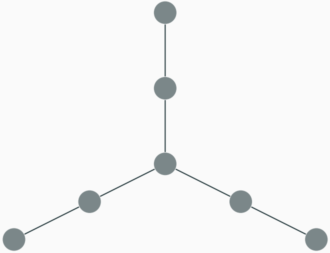


$$\mapsto \quad \left\{ \overbrace{\begin{matrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}} \right.$$

Bilabelled Graphs and Homomorphism Matrices



$$\mapsto \left\{ \overbrace{\begin{matrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}} \right.$$



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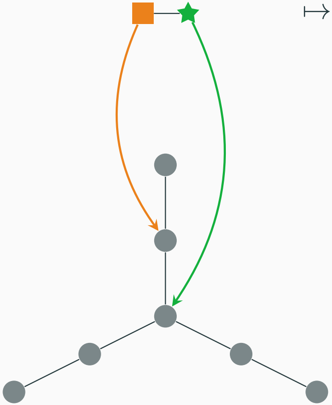
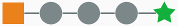


$\mapsto$

$$\left\{\overbrace{\begin{matrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}}\right.$$

Combinatorial and Algebraic Operations: Gluing+Unlabelling and Traces



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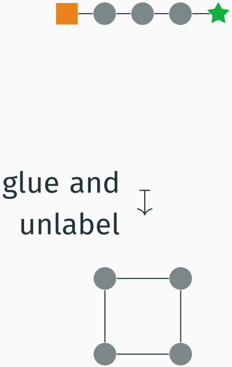

glue and unlabel

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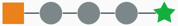
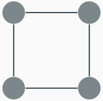


$$\mapsto \left\{ \overbrace{\begin{matrix} 12 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 6 & 0 & 5 & 0 & 5 & 0 \\ 4 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 5 & 0 & 6 & 0 & 5 & 0 \\ 4 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 5 & 0 & 5 & 0 & 6 & 0 \\ 4 & 0 & 1 & 0 & 1 & 0 & 2 \end{matrix}} \right.$$

glue and unlabel $\downarrow$



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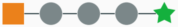
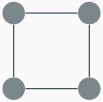


↦

$$\left\{\overbrace{\begin{matrix} 12 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 6 & 0 & 5 & 0 & 5 & 0 \\ 4 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 5 & 0 & 6 & 0 & 5 & 0 \\ 4 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 5 & 0 & 5 & 0 & 6 & 0 \\ 4 & 0 & 1 & 0 & 1 & 0 & 2 \end{matrix}}\right.$$

glue and unlabel ↧

↧ trace



↦

36

Examples: Trees, Paths, Cycles

1. Construct family $\mathcal{F}$ of **(bi)labelled graphs**

2. Define suitable **operations**

3. Prove that $\mathcal{F}$ is **finitely generated** under operations

4. Define **representation** and recover system of equations

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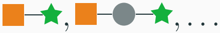

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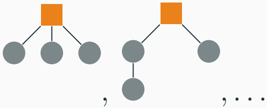

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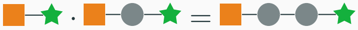

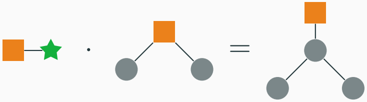

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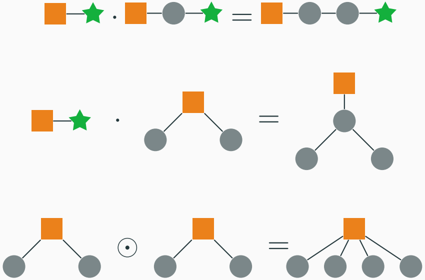

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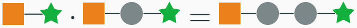

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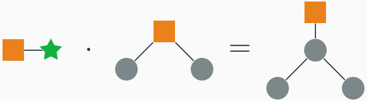

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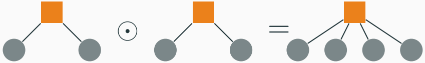

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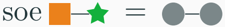

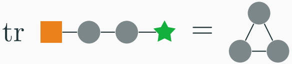

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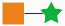
   - generator is 
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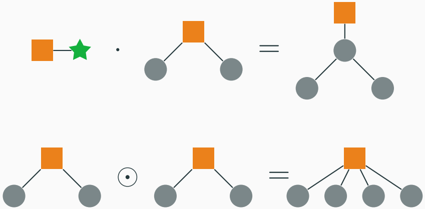

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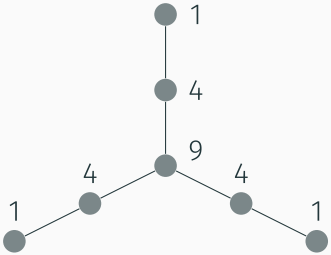
   - homomorphism vectors and matrices
   - missing ingredient: *variants of theorem by Specht and Wiegmann*



Specht–Wiegmann: Unitary, Pseudo-Stochastic, Doubly-Stochastic

When are complex square matrices $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ simultaneously similar?

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Theorem

X unitary

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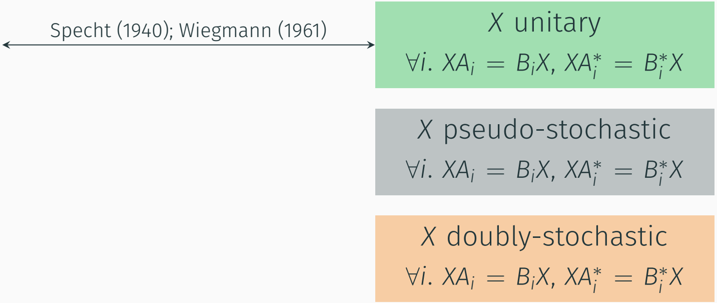

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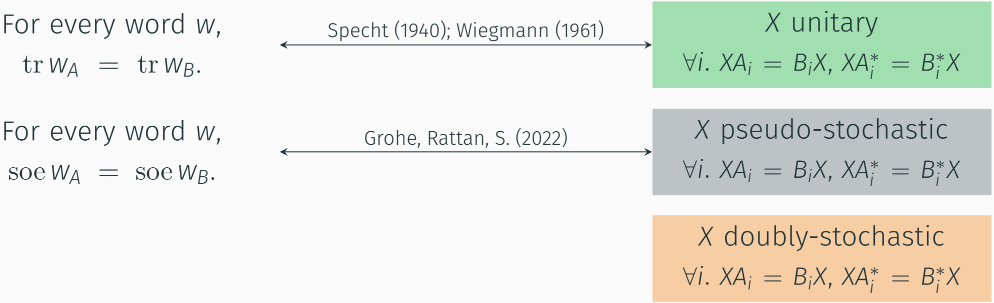

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Specht–Wiegmann: Words

Let Γ be the set of finite words over $\{x_1, \ldots, x_n, x_1^*, \ldots, x_n^*\}$.

Γ forms an *involution monoid.*

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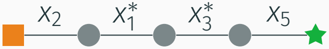

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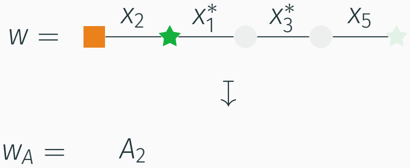

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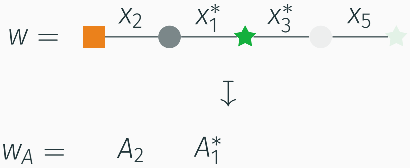

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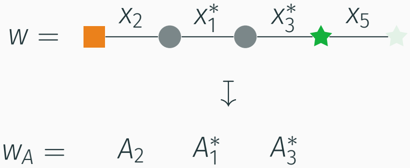

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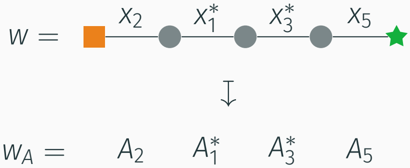

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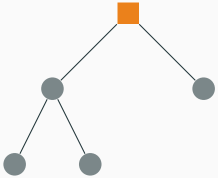

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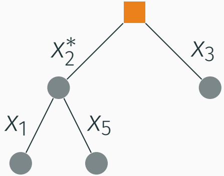

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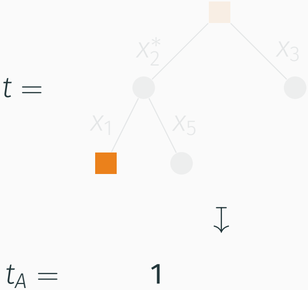

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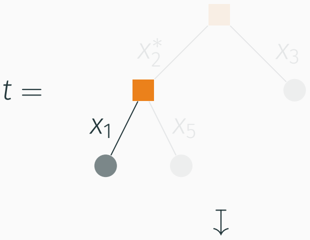

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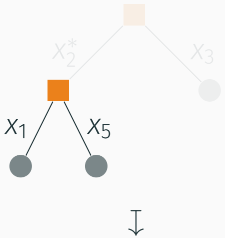

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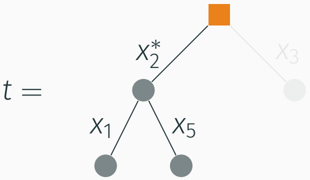

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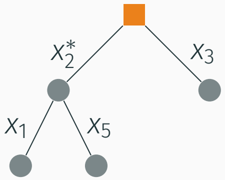

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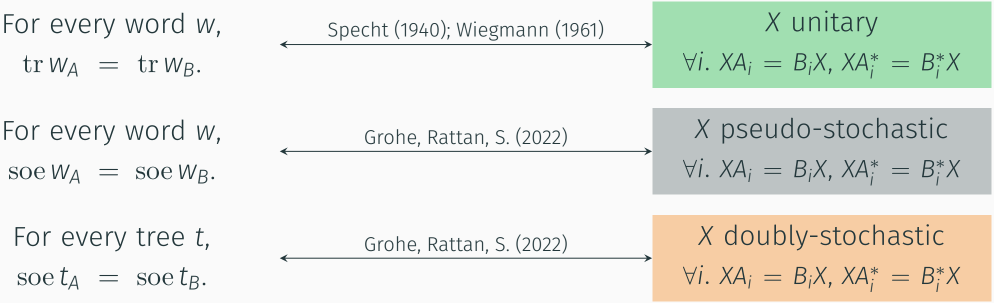

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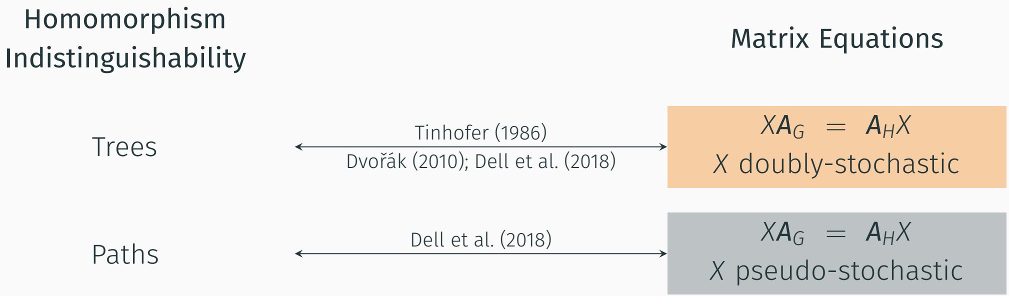
Graphs of Bounded Pathwidth and Sherali–Adams Relaxation



Homomorphism Indistinguishability

Matrix Equations

Trees

Tinhofer (1986)

Dvořák (2010); Dell et al. (2018)

$$XA_G = A_HX$$

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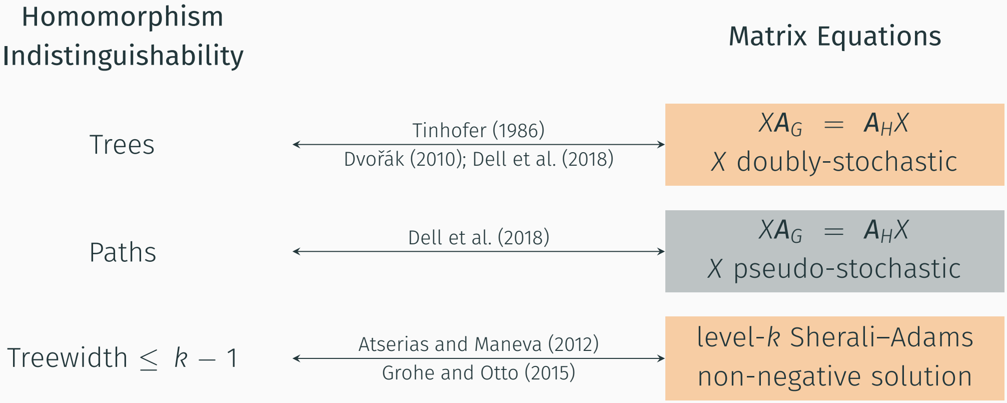

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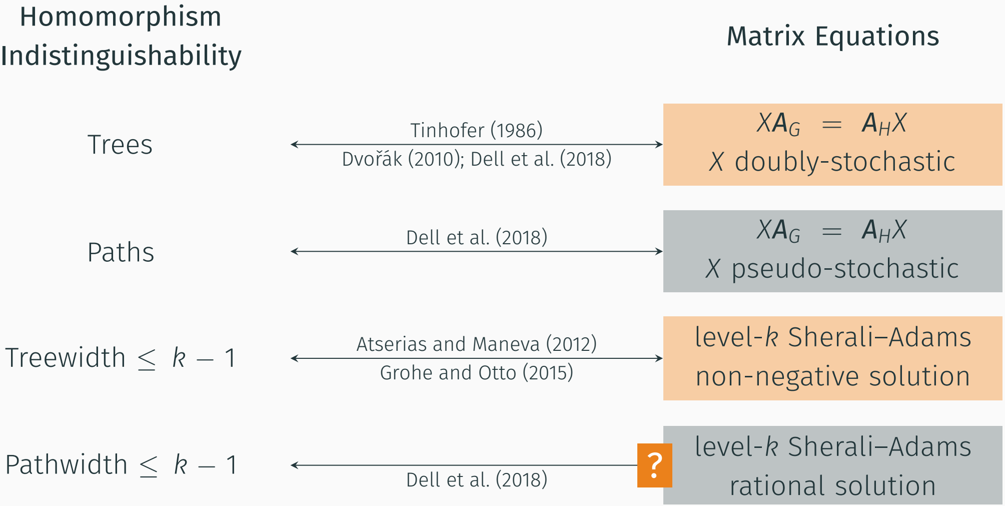

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Graphs of Bounded Pathwidth and Sherali–Adams Relaxation

Homomorphism Indistinguishability		Matrix Equations
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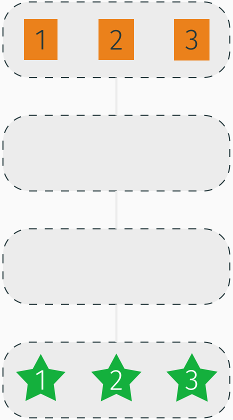

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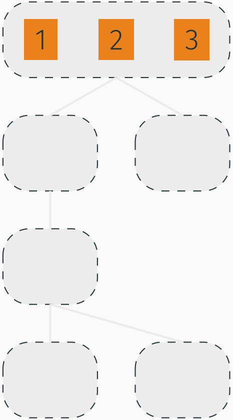

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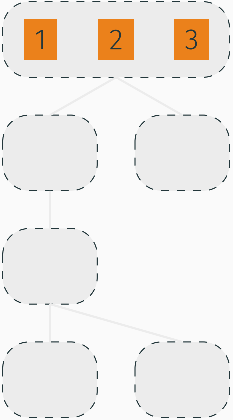

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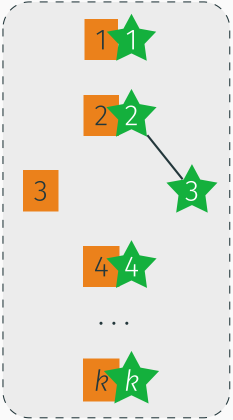

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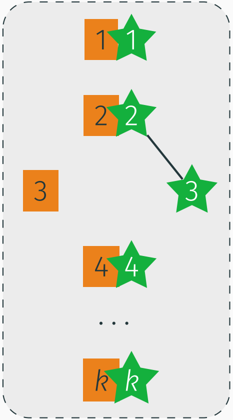

Limitations: Warped Wheel

The pieces **labelling**, **operations**, **finite generation**, and **representation** have to fit together.

Limitations: Warped Wheel

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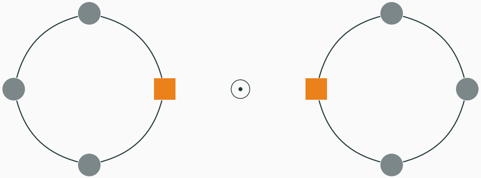

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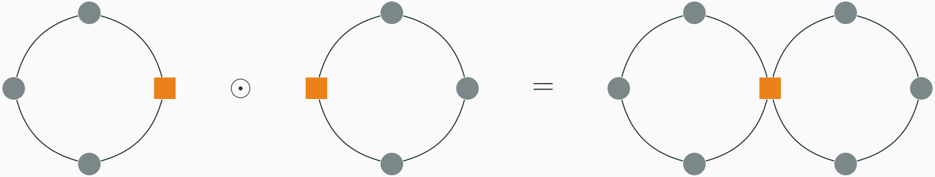

A Comonadic Strategy

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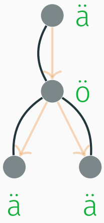

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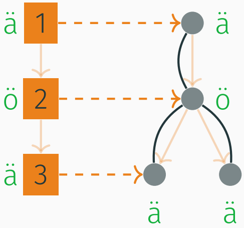

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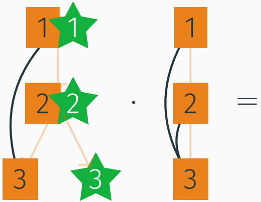

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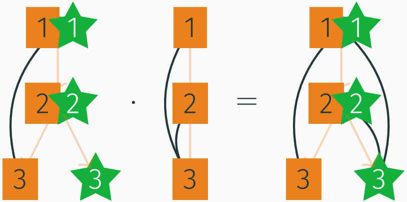

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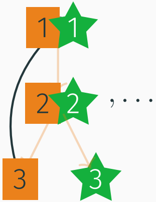

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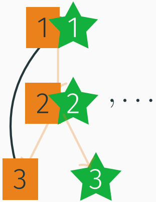
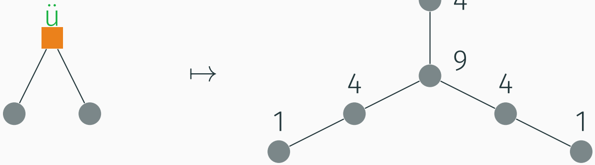


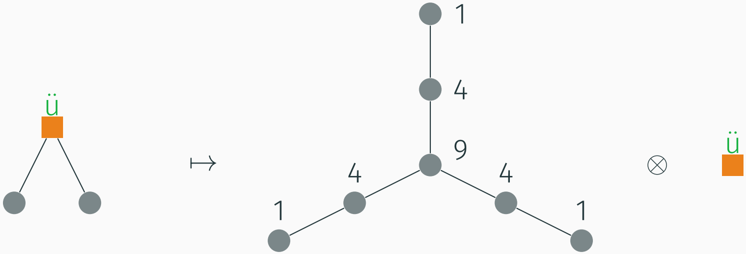
Augmented Homomorphism Representation

$$\mathcal{F} \longrightarrow \mathbb{C}^{V(G)}$$

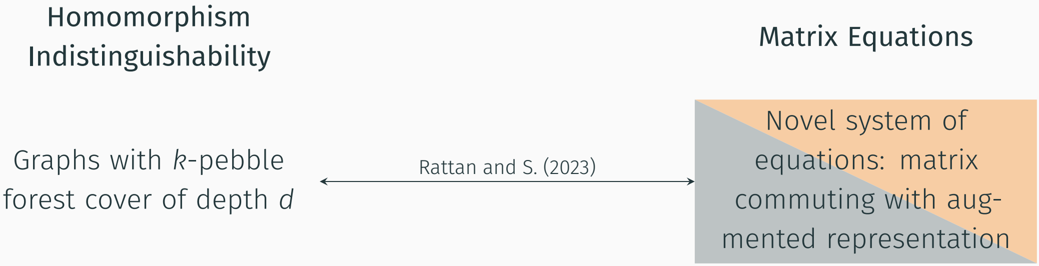


Augmented Homomorphism Representation

$$\mathcal{F} \longrightarrow \mathbb{C}^{V(G)} \otimes \bigoplus_{(L,\lambda)} \mathbb{C}$$



Graphs admitting k-pebble forest covers of depth d



Homomorphism Indistinguishability

Matrix Equations

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Rattan and S. (2023)

Novel system of equations: matrix commuting with augmented representation

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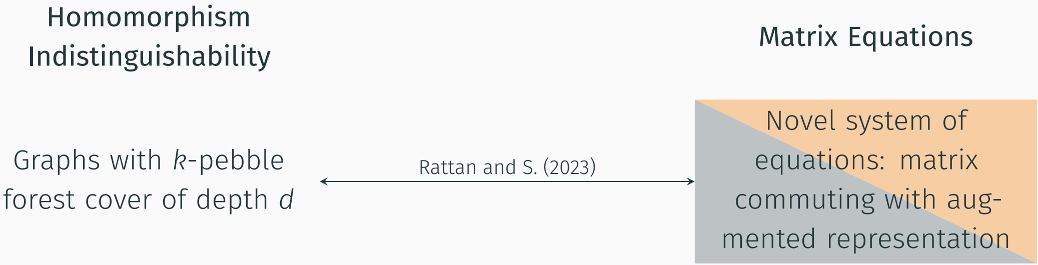

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This characterises *logical equivalence over* $\mathsf{C}_k \cap \mathsf{C}^d$, and with some modifications indistinguishability after *d rounds of the k-dimensional Weisfeiler–Leman algorithm*.

Towards a Theory of Homomorphism Indistinguishability

G and H are isomorphic iff
integer program $\mathrm{ISO}(G,H)$ is feasible

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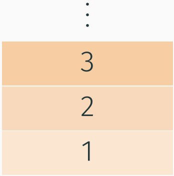
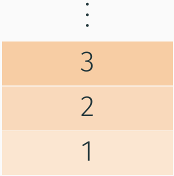

⋮
3
2
1

Lasserre
semidefinite prog.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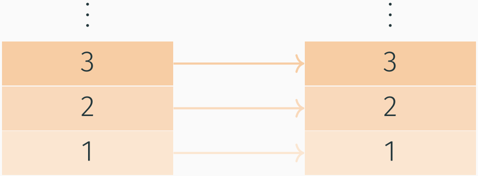

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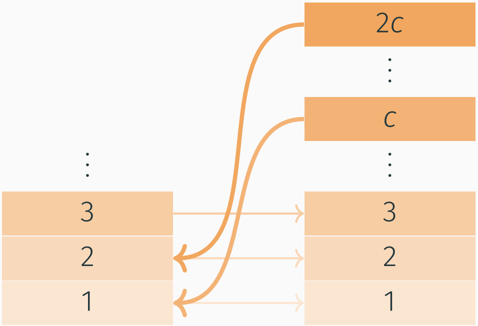

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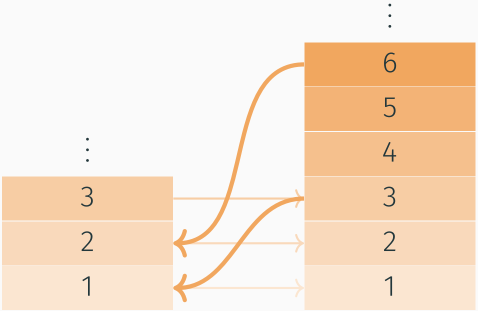

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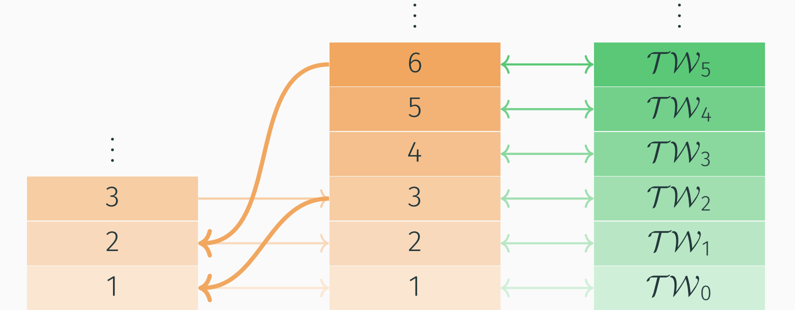

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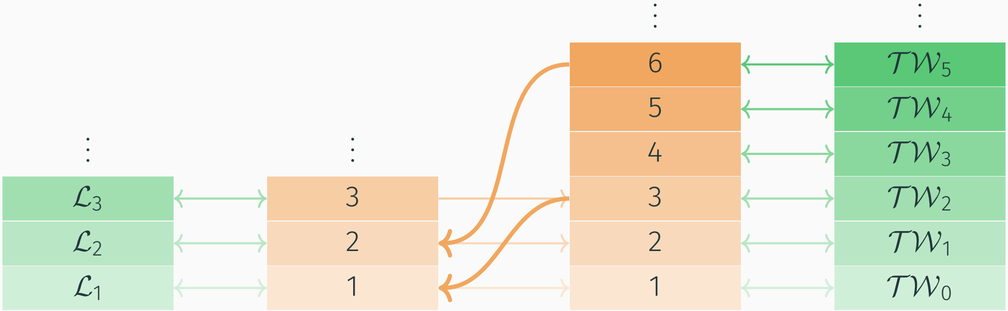

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From Equations to Graphs

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homomorphism tensors,

algebraic operations

Graph Class

(bi)labelled graphs,

combinatorial operations

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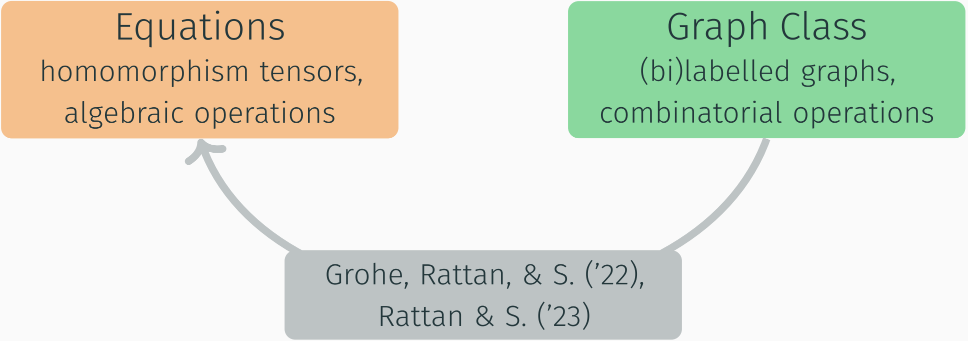

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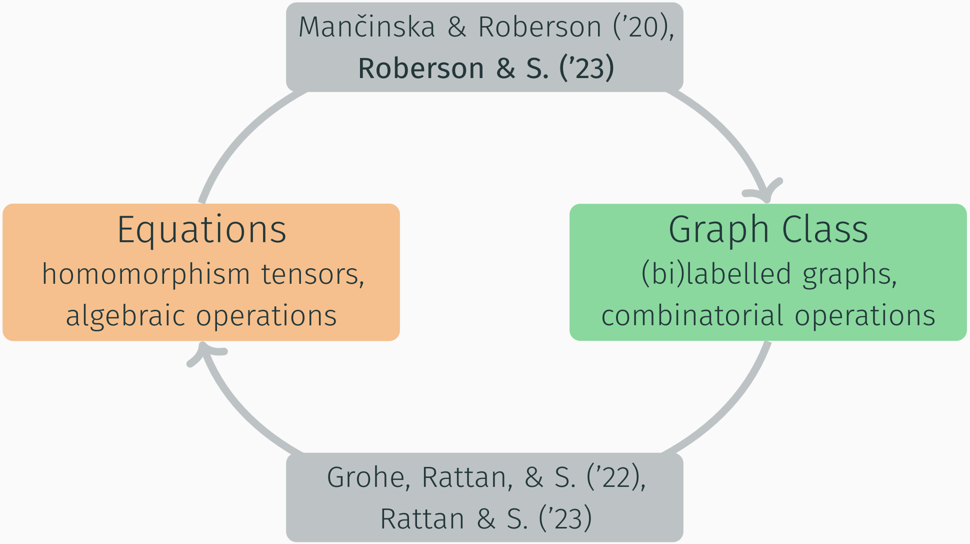

Mančinska & Roberson ('20),

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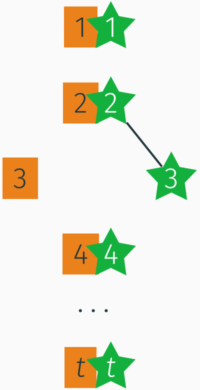
The Graph Class $\mathcal{L}_t$

A (t, t)-bilabelled graph is *atomic* if all its vertices are labelled.



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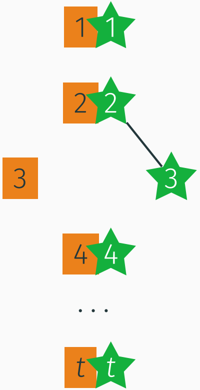
The class $\mathcal{L}_t$ is generated by atomic graphs under

- series composition,
- parallel composition with atomic graphs,
- permutation of labels.



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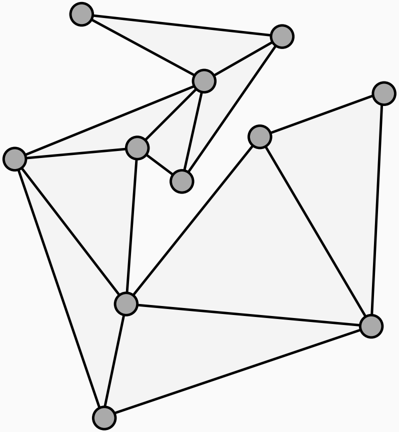
- $\mathcal{L}_t$ is minor-closed,
- $\mathcal{L}_1$ is the class of all outerplanar graphs.



Syntax and Semantics: Roberson's Conjecture

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Corollary (Roberson and S. (2023))

For every $t \geq 1$, there are graphs G and H such that $G \simeq^{\mathsf{SA}}_{3t-1} H$ and $G \not\simeq^{\mathsf{L}}_{t} H$.

Games for Roberson's Conjecture

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Question

Can game comonads yield more such results?

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The $\hom(F, -)$-functor maps products to products.

In the language of Marsden, Jakl, Shah (2023): There is a Kleisli law for the product functor $(G, H) \mapsto G \times H$.

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minors	*complements*	$G \mapsto \overline{G}$
summands	*disjoint unions*	$(G, H) \mapsto G + H$
subgraphs	*full complements*	$G \mapsto \widehat{G}$
induced subgraphs	*left lexicographic products*	$H \mapsto G[H]$ *for every* G
contracting edges	*right lexicographic products*	$G \mapsto G[H]$ *for every* H.

Logic meets Minors

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Corollary (Atserias et al. (2021))

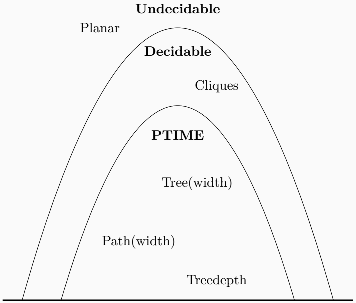
$\equiv_{\mathsf{FO}_k}$ *is not a homomorphism indistinguishability relation.*

Open Questions

Open Questions I

What is the complexity of deciding whether G and H are homomorphism indistinguishable over $\mathcal{F}$?

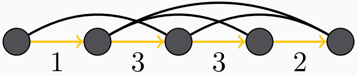
- Succinct matrix equations yield algorithms
- …may be used to prove undecidability



Open Questions II

Can matrix equations be cooked up for other graph classes?

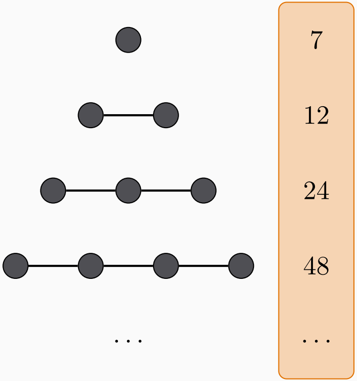
- path-like or tree-like graph classes, e.g. bounded cutwidth
- with comonadic strategy, only finite generation seems to be an issue



Open Questions III

When is a function $h\colon \mathcal{F} \to \mathbb{N}$ such that $h = \hom(-, H)$ for some graph H?

- Lovász and Schrijver (2009) answer this for $\mathcal{F} = \{\text{all graphs}\}$ using algebras of labelled graphs
- Applications in reconstruction



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$\dots$

Lovász and Schrijver (2009)

Let $\mathcal{C}$ be a category such that

- $\mathcal{C}$ is locally finite,
- $\mathcal{C}$ has pushouts and an initial object 0,
- every morphism is the product of an epimorphism and a monomorphism,
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Then $h \colon \operatorname{obj} \mathcal{C} \to \mathbb{R}$ is of the form $h = \hom(-, H)$ if and only if

- $h(0) = 1$,
- h is multiplicative over coproducts,
- the matrix $N(h, L)$ is positive semidefinite for every L.

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- every morphism is the product of an epimorphism and a monomorphism,
- there is a *generator* $G \in \operatorname{obj} \mathcal{C}$, i.e. $\forall F \exists n \in \mathbb{N}.\ nG \twoheadrightarrow F$.

Then $h\colon \operatorname{obj} \mathcal{C} \to \mathbb{R}$ is of the form $h = \hom(-, H)$ if and only if

- $h(0) = 1$,
- h is multiplicative over coproducts,
- the matrix $N(h, L)$ is positive semidefinite for every L.

Question

Characterise $h\colon \operatorname{im} U^{\mathbb{C}} \to \mathbb{R}$ *of the form* $h = \hom_{\Sigma}(-, H) = \hom_{\mathrm{EM}(\mathbb{C})}(-, F^{\mathbb{C}} H)$.

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- **Check out Grohe et al. (2022); Rattan and Seppelt (2023); Roberson and Seppelt (2023); Seppelt (2023)**!

References

Atserias, A., Kolaitis, P. G., and Wu, W.-L. (2021). On the Expressive Power of Homomorphism Counts. In *36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021*, pages 1–13.

Atserias, A. and Maneva, E. (2012). Sherali–Adams Relaxations and Indistinguishability in Counting Logics. In *Proceedings of the 3rd Innovations in Theoretical Computer Science Conference*, ITCS '12, pages 367–379, New York, NY, USA. Association for Computing Machinery.

Bibliography ii

Atserias, A. and Ochremiak, J. (2018). Definable ellipsoid method, sums-of-squares proofs, and the isomorphism problem. In Dawar, A. and Grädel, E., editors, *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018, Oxford, UK, July 09-12, 2018*, pages 66–75. ACM.

Dawar, A., Jakl, T., and Reggio, L. (2021). Lovász-type theorems and game comonads. In *36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021*, pages 1–13. IEEE.

Dell, H., Grohe, M., and Rattan, G. (2018). Lovász Meets Weisfeiler and Leman. In Chatzigiannakis, I., Kaklamanis, C., Marx, D., and Sannella, D., editors, *45th International Colloquium on Automata, Languages, and Programming, ICALP 2018, July 9-13, 2018, Prague, Czech Republic*, volume 107 of *LIPIcs*, pages 40:1–40:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik.

Dvořák, Z. (2010). On recognizing graphs by numbers of homomorphisms. *Journal of Graph Theory*, 64(4):330–342.

Bibliography iii

Grohe, M. (2020). word2vec, node2vec, graph2vec, x2vec: Towards a theory of vector embeddings of structured data. In Suciu, D., Tao, Y., and Wei, Z., editors, *Proceedings of the 39th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, PODS 2020, Portland, OR, USA, June 14-19, 2020*, pages 1–16. ACM.

Grohe, M. and Otto, M. (2015). Pebble Games and Linear Equations. *J. Symb. Log.*, 80(3):797–844.

Grohe, M., Rattan, G., and Seppelt, T. (2022). Homomorphism Tensors and Linear Equations. In Bojańczyk, M., Merelli, E., and Woodruff, D. P., editors, *49th International Colloquium on Automata, Languages, and Programming (ICALP 2022)*, volume 229 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 70:1–70:20, Dagstuhl, Germany. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. ISSN: 1868-8969.

Lovász, L. (1967). Operations with structures. *Acta Mathematica Academiae Scientiarum Hungarica*, 18(3):321–328.

Bibliography iv

Lovász, L. and Schrijver, A. (2009). Semidefinite Functions on Categories. *Electron. J. Comb.*, 16(2).

Mančinska, L. and Roberson, D. E. (2020). Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs. In *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*, pages 661–672.

Neuen, D. (2023). Homomorphism-Distinguishing Closedness for Graphs of Bounded Tree-Width. arXiv:2304.07011 [cs, math].

Rattan, G. and Seppelt, T. (2023). Weisfeiler-Leman and Graph Spectra. In *Proceedings of the 2023 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2268–2285. _eprint: https://epubs.siam.org/doi/pdf/10.1137/1.9781611977554.ch87.

Roberson, D. E. (2022). Oddomorphisms and homomorphism indistinguishability over graphs of bounded degree. Number: arXiv:2206.10321.

Bibliography v

Roberson, D. E. and Seppelt, T. (2023). Lasserre Hierarchy for Graph Isomorphism and Homomorphism Indistinguishability. arXiv:2302.10538 [cs, math].

Seppelt, T. (2023). Logical Equivalences, Homomorphism Indistinguishability, and Forbidden Minors. arXiv:2302.11290 [cs, math].

Specht, W. (1940). Zur Theorie der Matrizen. II. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 50:19–23.

Tinhofer, G. (1986). Graph isomorphism and theorems of Birkhoff type. *Computing*, 36(4):285–300.

Wiegmann, N. A. (1961). Necessary and sufficient conditions for unitary similarity. *Journal of the Australian Mathematical Society*, 2(1):122–126. Edition: 2009/04/09 Publisher: Cambridge University Press.

Bibliography vi

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