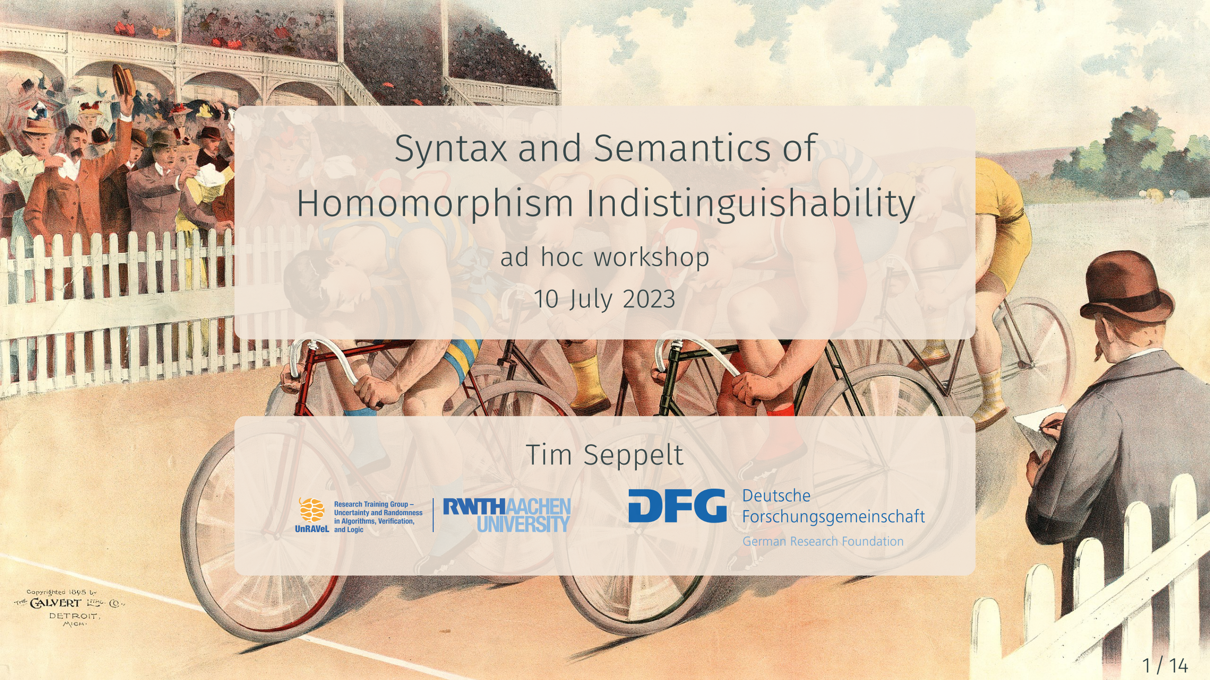

Syntax and Semantics of Homomorphism Indistinguishability

ad hoc workshop

10 July 2023

Tim Seppelt

UnRAVeL Research Training Group – Uncertainty and Randomness in Algorithms, Verification, and Logic
RWTH AACHEN UNIVERSITY
DFG Deutsche Forschungsgemeinschaft
German Research Foundation

Homomorphism Indistinguishability

graph class $\mathcal{F}$	**relation $\equiv_{\mathcal{F}}$**	
all graphs	isomorphism	Lovász (1967)
planar graphs	quantum isomorphism	Mančinska and Roberson (2020)
treewidth $\leq k$	C^{k+1}-equivalence	Dvořák (2010)
treedepth $\leq d$	C_d-equivalence	Grohe (2020)
…	…	

When is an equivalence relation between graphs a homomorphism indistinguishability relation?

Properties of Homomorphism Indistinguishability Relations

Observation ($\equiv_{\mathcal{F}}$ is preserved under categorical products)

If $G_1 \equiv_{\mathcal{F}} H_1$ and $G_2 \equiv_{\mathcal{F}} H_2$ then $G_1 \times G_2 \equiv_{\mathcal{F}} H_1 \times H_2$.

Properties of Homomorphism Indistinguishability Relations

Observation ($\equiv_{\mathcal{F}}$ is preserved under categorical products)

If $G_1 \equiv_{\mathcal{F}} H_1$ and $G_2 \equiv_{\mathcal{F}} H_2$ then $G_1 \times G_2 \equiv_{\mathcal{F}} H_1 \times H_2$.

For every graph F,

$$\hom(F, G_1 \times G_2) = \hom(F, G_1) \hom(F, G_2).$$

Closure properties of $\mathcal{F}$ correspond to preservation properties of $\equiv_{\mathcal{F}}$.

Properties of Homomorphism Indistinguishability Relations

Closure properties of $\mathcal{F}$ correspond to preservation properties of $\equiv_{\mathcal{F}}$.

Theorem (S. (2023))

For every homomorphism distinguishing closed graph class $\mathcal{F}$, tfae:

$\mathcal{F}$ is closed under minors	***$\equiv_{\mathcal{F}}$ is preserved under complements***	$G \mapsto \overline{G}$

Properties of Homomorphism Indistinguishability Relations

Closure properties of $\mathcal{F}$ correspond to preservation properties of $\equiv_{\mathcal{F}}$.

Theorem (S. (2023))

For every homomorphism distinguishing closed graph class $\mathcal{F}$, tfae:

$\mathcal{F}$ ***is closed under***	$\equiv_{\mathcal{F}}$ ***is preserved under***	
minors	*complements*	$G \mapsto \overline{G}$
summands	*disjoint unions*	$(G, H) \mapsto G + H$

Properties of Homomorphism Indistinguishability Relations

Closure properties of $\mathcal{F}$ correspond to preservation properties of $\equiv_{\mathcal{F}}$.

Theorem (S. (2023))

For every homomorphism distinguishing closed graph class $\mathcal{F}$, tfae:

$\mathcal{F}$ ***is closed under***	$\equiv_{\mathcal{F}}$ ***is preserved under***	
minors	*complements*	$G \mapsto \overline{G}$
summands	*disjoint unions*	$(G, H) \mapsto G + H$
subgraphs	*full complements*	$G \mapsto \widehat{G}$
induced subgraphs	*left lexicographic products*	$H \mapsto G[H]$ *for every* G
contracting edges	*right lexicographic products*	$G \mapsto G[H]$ *for every* H.

Proof Idea

Claim: $\mathcal{F}$ closed under minors $\implies$ $\equiv_{\mathcal{F}}$ preserved under complements.

Proof Idea

Claim: $\mathcal{F}$ closed under minors $\implies$ $\equiv_{\mathcal{F}}$ preserved under complements.

Idea: Derive a formula for $\hom(F, \overline{G})$ in terms of $\hom(F', G)$ where F' is minor of F.

Proof Idea

Claim: $\mathcal{F}$ closed under minors $\implies$ $\equiv_{\mathcal{F}}$ preserved under complements.

Idea: Derive a formula for $\hom(F, \overline{G})$ in terms of $\hom(F', G)$ where F' is minor of F.

$$\hom(F, \widehat{G}) = \sum_{\substack{F' \subseteq F \\ V(F') = V(F)}} (-1)^{E(F')} \hom(F', G).$$

Proof Idea

Claim: $\mathcal{F}$ closed under minors $\implies$ $\equiv_{\mathcal{F}}$ preserved under complements.

Idea: Derive a formula for $\hom(F, \overline{G})$ in terms of $\hom(F', G)$ where F' is minor of F.

$$\hom(F, \widehat{G}) = \sum_{\substack{F' \subseteq F \\ V(F') = V(F)}} (-1)^{E(F')} \hom(F', G).$$

$$\hom(F, G^\circ) = \sum_{L \subseteq E(F)} \hom(F \oslash L, G).$$

Proof Idea

Claim: $\mathcal{F}$ closed under minors $\implies$ $\equiv_{\mathcal{F}}$ preserved under complements.

Idea: Derive a formula for $\hom(F, \overline{G})$ in terms of $\hom(F', G)$ where F' is minor of F.

$$\hom(F, \widehat{G}) = \sum_{\substack{F' \subseteq F \\ V(F') = V(F)}} (-1)^{E(F')} \hom(F', G).$$

$$\hom(F, G^{\circ}) = \sum_{L \subseteq E(F)} \hom(F \oslash L, G).$$

$$\hom(F, \overline{G}) = \sum_{\substack{F' \subseteq F \\ V(F') = V(F)}} (-1)^{|E(F')|} \sum_{L \subseteq E(F')} \hom(F' \oslash L, G).$$

Given a graph class $\mathcal{F}$,

$$\mathrm{cl}(\mathcal{F}) \coloneqq \{F \text{ graph} \mid \forall G, H.\ G \equiv_{\mathcal{F}} H \implies \hom(F, G) = \hom(F, H)\}.$$

Homomorphism Distinguishing Closure

Given a graph class $\mathcal{F}$,

$$\mathrm{cl}(\mathcal{F}) := \{F \text{ graph} \mid \forall G, H.\ G \equiv_{\mathcal{F}} H \implies \hom(F, G) = \hom(F, H)\}.$$

Largest graph class whose homomorphism indistinguishability relation coincides with $\equiv_{\mathcal{F}}$.

Given a graph class $\mathcal{F}$,

$$\mathrm{cl}(\mathcal{F}) := \{F \text{ graph} \mid \forall G, H.\ G \equiv_{\mathcal{F}} H \implies \hom(F, G) = \hom(F, H)\}.$$

Largest graph class whose homomorphism indistinguishability relation coincides with $\equiv_{\mathcal{F}}$.

Lemma (Curticapean et al. (2017))

Suppose for all G, H,

$$G \equiv_{\mathcal{F}} H \implies \sum_{i \in [n]} \alpha_i \hom(K_i, G) = \sum_{i \in [n]} \alpha_i \hom(K_i, H)$$

then $K_1, \dots, K_n \in \mathrm{cl}(\mathcal{F})$.

Proof Idea

Claim: $\equiv_{\mathcal{F}}$ preserved under complements $\implies$ $\mathrm{cl}(\mathcal{F})$ closed under minors.

$$\hom(F, \overline{G}) = \sum_{\substack{F' \subseteq F \\ V(F') = V(F)}} (-1)^{|E(F')|} \sum_{L \subseteq E(F')} \hom(F' \oslash L, G).$$

Coefficients of all minors of F are non-zero.

Self-complementary Logics

Theorem (S. (2023))

For every homomorphism distinguishing closed graph class $\mathcal{F}$, tfae:

***$\mathcal{F}$ is closed under** minors*	***$\equiv_{\mathcal{F}}$ is preserved under** complements*

- **Feasibility of integer programming relaxations for graph isomorphism**
  Graphs are encoded via atomic types of vertex tuples

Self-complementary Logics

Theorem (S. (2023))

For every homomorphism distinguishing closed graph class $\mathcal{F}$, tfae:

$\mathcal{F}$ is closed under minors

$\equiv_{\mathcal{F}}$ is preserved under complements

- **Feasibility of integer programming relaxations for graph isomorphism**
  Graphs are encoded via atomic types of vertex tuples
- ***Self-complementary logics*** $(\mathsf{L}, \models)$
  For every sentence $\varphi \in \mathsf{L}$, there is $\overline{\varphi} \in \mathsf{L}$ such that $G \models \varphi \iff \overline{G} \models \overline{\varphi}$.
  E.g., replace Exy by $\neg Exy \wedge (x \neq y)$.

Ruling out Homomorphism Indistinguishability Relations

Corollary (Atserias et al. (2021))

FO^k*-equivalence is not a homomorphism indistinguishability relation.*

Ruling out Homomorphism Indistinguishability Relations

Corollary (Atserias et al. (2021))

FO^k*-equivalence is not a homomorphism indistinguishability relation.*

FO^k is self-complementary.

Ruling out Homomorphism Indistinguishability Relations

Corollary (Atserias et al. (2021))

FO^k*-equivalence is not a homomorphism indistinguishability relation.*

FO^k is self-complementary.

Suppose $\equiv_{\mathcal{F}}$ characterises FO^k-equivalence. Wlog $\mathcal{F}$ is minor-closed.

Ruling out Homomorphism Indistinguishability Relations

Corollary (Atserias et al. (2021))

FO^k*-equivalence is not a homomorphism indistinguishability relation.*

FO^k is self-complementary.

Suppose $\equiv_{\mathcal{F}}$ characterises FO^k-equivalence. Wlog $\mathcal{F}$ is minor-closed.

$K_k \equiv_{\mathsf{FO}^k} K_{k+1}$ but $\hom(K_1, K_k) \neq \hom(K_1, K_{k+1})$, so $K_1 \notin \mathcal{F}$, contradiction!

Graph Minor Theory rules out Homomorphism Indistinguishability

Theorem (Robertson and Seymour (1986))

For a minor-closed graph class $\mathcal{F}$, tfae:

- *$\mathcal{F}$ has unbounded treewidth,*
- *$\mathcal{F}$ contains all planar graphs.*

Graph Minor Theory rules out Homomorphism Indistinguishability

Theorem (Robertson and Seymour (1986))

For a minor-closed graph class $\mathcal{F}$, tfae:

- *$\mathcal{F}$ has unbounded treewidth,*
- *$\mathcal{F}$ contains all planar graphs.*

Corollary (S. (2023))

Let L *be a self-complementary logic. Suppose that*

- L*-equivalence is homomorphism indistinguishability relation,*

Graph Minor Theory rules out Homomorphism Indistinguishability

Theorem (Robertson and Seymour (1986))

For a minor-closed graph class $\mathcal{F}$, tfae:

- $\mathcal{F}$ *has unbounded treewidth,*
- $\mathcal{F}$ *contains all planar graphs.*

Corollary (S. (2023))

Let L *be a self-complementary logic. Suppose that*

- L*-equivalence is homomorphism indistinguishability relation,*
- *for every* $k \in \mathbb{N}$*, there exist graphs* G *and* H *such that* $G \equiv_{\mathsf{C}^k} H$ *and* $G \not\equiv_{\mathsf{L}} H$*.*

Graph Minor Theory rules out Homomorphism Indistinguishability

Theorem (Robertson and Seymour (1986))

For a minor-closed graph class $\mathcal{F}$, tfae:

- *$\mathcal{F}$ has unbounded treewidth,*
- *$\mathcal{F}$ contains all planar graphs.*

Corollary (S. (2023))

Let L be a self-complementary logic. Suppose that

- *L-equivalence is homomorphism indistinguishability relation,*
- *for every $k \in \mathbb{N}$, there exist graphs G and H such that $G \equiv_{\mathsf{C}^k} H$ and $G \not\equiv_{\mathsf{L}} H$.*

Then all L-equivalent graphs are quantum isomorphic.

Outlook: Logics stronger than Weisfeiler–Leman

Theorem (Lichter, Pago, S. (2023+))

$\mathrm{LA}^k(\mathbb{Q})$*-equivalence is not a homomorphism indistinguishable relation.*

Outlook: Logics stronger than Weisfeiler–Leman

Theorem (Lichter, Pago, S. (2023+))

$\mathrm{LA}^k(Q)$*-equivalence is not a homomorphism indistinguishable relation.*

$\mathrm{LA}^k(Q)$ does not distinguish CFI-like graphs over some planar base graph.

Outlook: Logics stronger than Weisfeiler–Leman

Theorem (Lichter, Pago, S. (2023+))

$\mathrm{LA}^k(Q)$*-equivalence is not a homomorphism indistinguishable relation.*

$\mathrm{LA}^k(Q)$ does not distinguish CFI-like graphs over some planar base graph.

Roberson (2022): CFI-like graphs over planar base graph are not quantum isomorphic

Polymorphisms

Theorem

$G \mapsto \overline{G}$

$(G, H) \mapsto G + H$

$G \mapsto \widehat{G}$

$H \mapsto G[H]$ *for every* G

$G \mapsto G[H]$ *for every* H.

Polymorphisms

A *polymorphism* of a relation $\equiv$ is a map $\mathfrak{f}$ such that

$$G_1 \equiv H_1, \ldots, G_k \equiv H_k \implies \mathfrak{f}(G_1, \ldots, G_k) \equiv \mathfrak{f}(H_1, \ldots, H_k).$$

Theorem

$G \mapsto \overline{G}$

$(G, H) \mapsto G + H$

$G \mapsto \widehat{G}$

$H \mapsto G[H]$ *for every* G

$G \mapsto G[H]$ *for every* H.

Polymorphisms

A *polymorphism* of a relation $\equiv$ is a map $\mathfrak{f}$ such that

$$G_1 \equiv H_1, \ldots, G_k \equiv H_k \implies \mathfrak{f}(G_1, \ldots, G_k) \equiv \mathfrak{f}(H_1, \ldots, H_k).$$

Closure properties of $\mathcal{F}$ are reflected by the clone of polymorphisms of $\equiv_{\mathcal{F}}$.

Theorem

$G \mapsto \overline{G}$

$(G, H) \mapsto G + H$

$G \mapsto \widehat{G}$

$H \mapsto G[H]$ *for every* G

$G \mapsto G[H]$ *for every* H.

Polymorphisms

A *polymorphism* of a relation $\equiv$ is a map $\mathfrak{f}$ such that

$$G_1 \equiv H_1, \ldots, G_k \equiv H_k \implies \mathfrak{f}(G_1, \ldots, G_k) \equiv \mathfrak{f}(H_1, \ldots, H_k).$$

Closure properties of $\mathcal{F}$ are reflected by the clone of polymorphisms of $\equiv_{\mathcal{F}}$.

What about

- complexity of HomInd($\mathcal{F}$)?
- treewidth of $\mathcal{F}$?

Theorem

$G \mapsto \overline{G}$

$(G, H) \mapsto G + H$

$G \mapsto \widehat{G}$

$H \mapsto G[H]$ *for every* G

$G \mapsto G[H]$ *for every* H.

Conclusion

- Closure properties of $\mathcal{F}$ correspond to preservation properties of $\equiv_{\mathcal{F}}$.

Conclusion

- Closure properties of $\mathcal{F}$ correspond to preservation properties of $\equiv_{\mathcal{F}}$.
- Self-complementary logics have homomorphism indistinguishability characterisations over minor-closed graph classes (if at all).

Conclusion

- Closure properties of $\mathcal{F}$ correspond to preservation properties of $\equiv_{\mathcal{F}}$.
- Self-complementary logics have homomorphism indistinguishability characterisations over minor-closed graph classes (if at all).
- When is an equivalence relation between graphs a homomorphism indistinguishability relation?

Conclusion

- Closure properties of $\mathcal{F}$ correspond to preservation properties of $\equiv_{\mathcal{F}}$.
- Self-complementary logics have homomorphism indistinguishability characterisations over minor-closed graph classes (if at all).
- When is an equivalence relation between graphs a homomorphism indistinguishability relation?
- Check out `arXiv:2302.11290`!

References

Atserias, A., Kolaitis, P. G., and Wu, W.-L. (2021). On the Expressive Power of Homomorphism Counts. In *36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021*, pages 1–13.

Curticapean, R., Dell, H., and Marx, D. (2017). Homomorphisms Are a Good Basis for Counting Small Subgraphs. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2017, pages 210–223, New York, NY, USA. Association for Computing Machinery. event-place: Montreal, Canada.

Dvořák, Z. (2010). On recognizing graphs by numbers of homomorphisms. *Journal of Graph Theory*, 64(4):330–342.

Grohe, M. (2020). Counting Bounded Tree Depth Homomorphisms. In *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '20, pages 507–520, New York, NY, USA. Association for Computing Machinery. event-place: Saarbrücken, Germany.

Lovász, L. (1967). Operations with structures. *Acta Mathematica Academiae Scientiarum Hungarica*, 18(3):321–328.

Mančinska, L. and Roberson, D. E. (2020). Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs. In *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*, pages 661–672.

Roberson, D. E. (2022). Oddomorphisms and homomorphism indistinguishability over graphs of bounded degree.

Robertson, N. and Seymour, P. (1986). Graph minors. V. Excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 41(1):92–114.

Bibliography iii

Pictures:

- "Bicycle race scene. A peloton of six cyclists crosses the finish line in front of a crowded grandstand, observed by a referee." (1895) by Calvert Lithographic Co., Detroit, Michigan, Public Domain, via Wikimedia Commons. `https://commons.wikimedia.org/wiki/File:Bicycle_race_scene,_1895.jpg`
- Encyclopedic manuscript containing allegorical and medical drawings, `http://lccn.loc.gov/50041709`, Public domain, via Wikimedia Commons