Lasserre Hierarchy for Graph Isomorphism and Homomorphism Indistinguishability

ICALP 2023

David E. Roberson and Tim Seppelt

UnRAVeL Research Training Group – Uncertainty and Randomness in Algorithms, Verification, and Logic
RWTH AACHEN UNIVERSITY
DFG Deutsche Forschungsgemeinschaft
German Research Foundation

G and H are isomorphic iff
integer program $\mathrm{ISO}(G, H)$ is feasible

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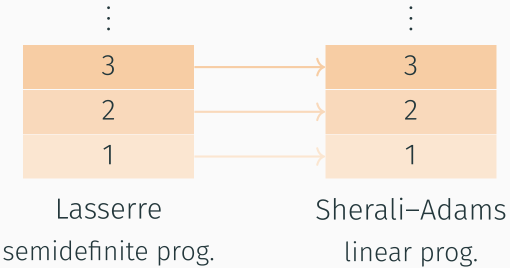
Lasserre	Sherali–Adams
⋮	⋮
3	3
2	2
1	1
Lasserre	Sherali–Adams
semidefinite prog.	linear prog.

G and H are isomorphic iff
integer program $\mathrm{ISO}(G, H)$ is feasible



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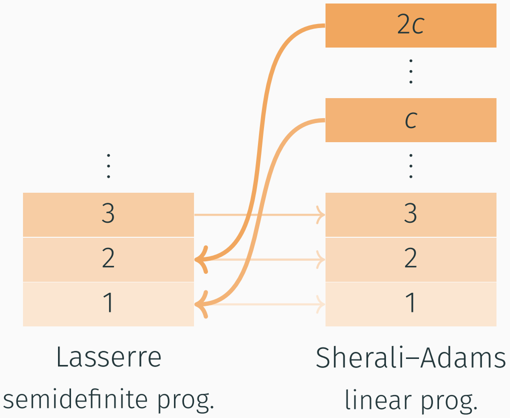

Atserias and Ochremiak (2018)

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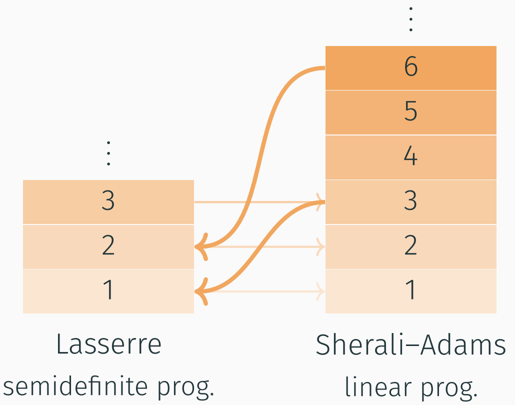
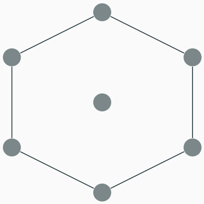
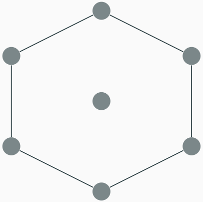
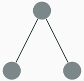
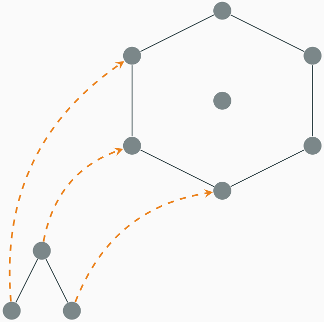


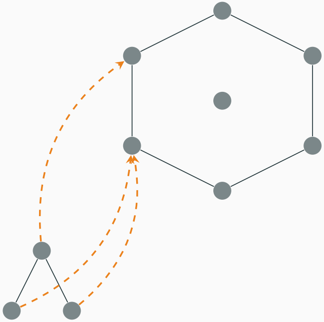
Atserias and Ochremiak (2018), **Roberson & S. (2023)**

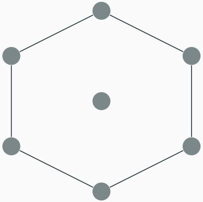


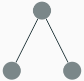












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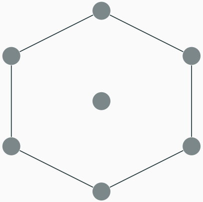

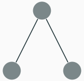

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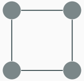

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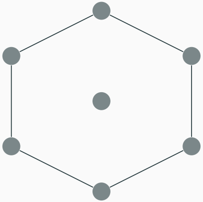
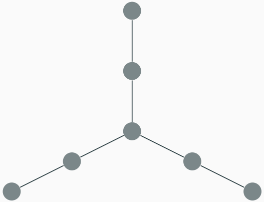
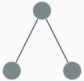
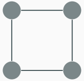
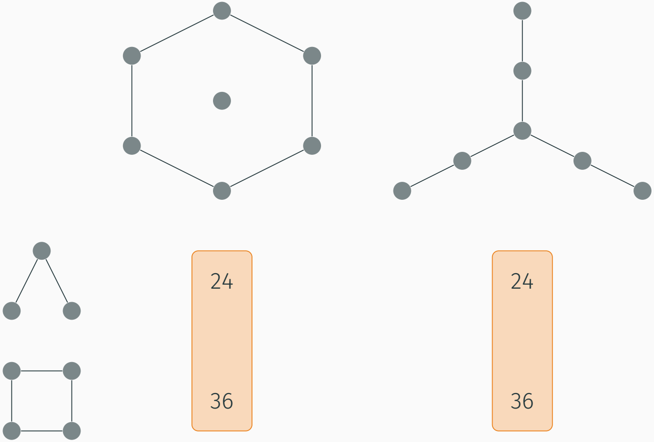
		
	24	24
	36	36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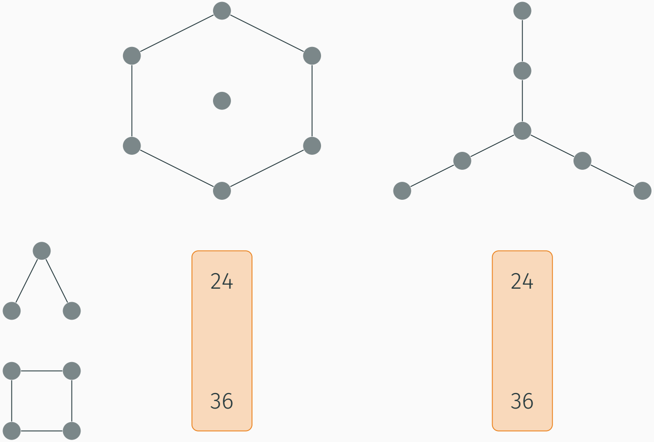
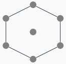
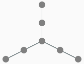


The graphs  and  are **homomorphism indistinguishable** over { ,  }.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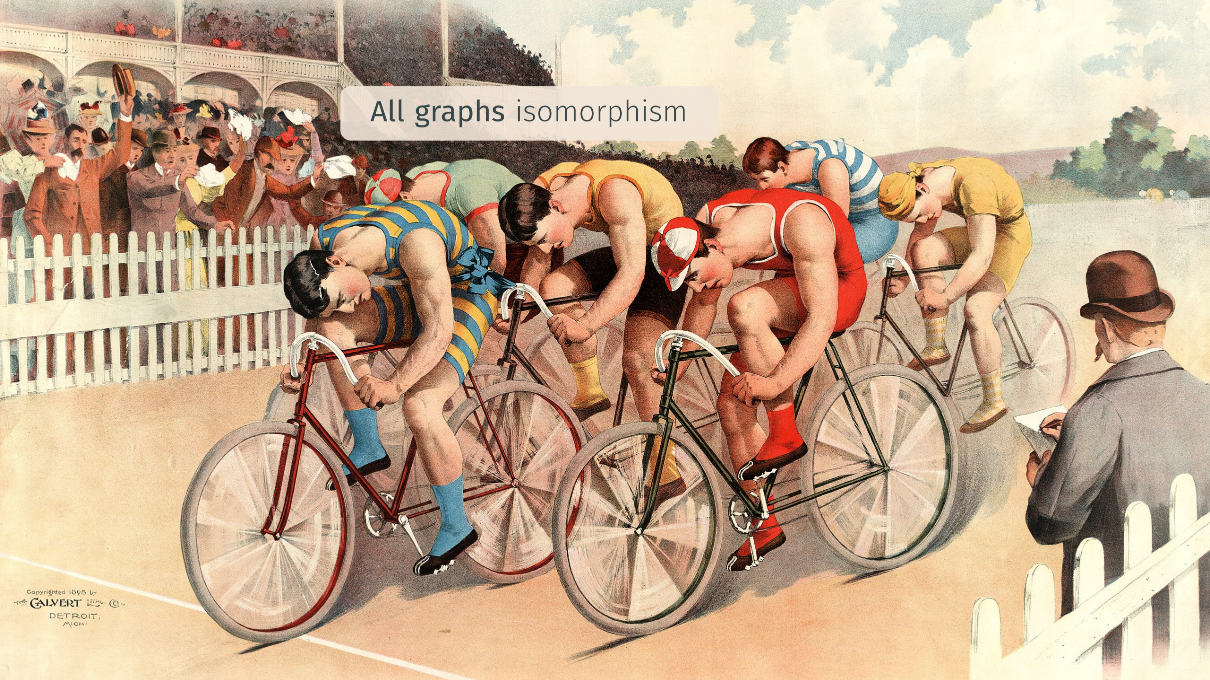
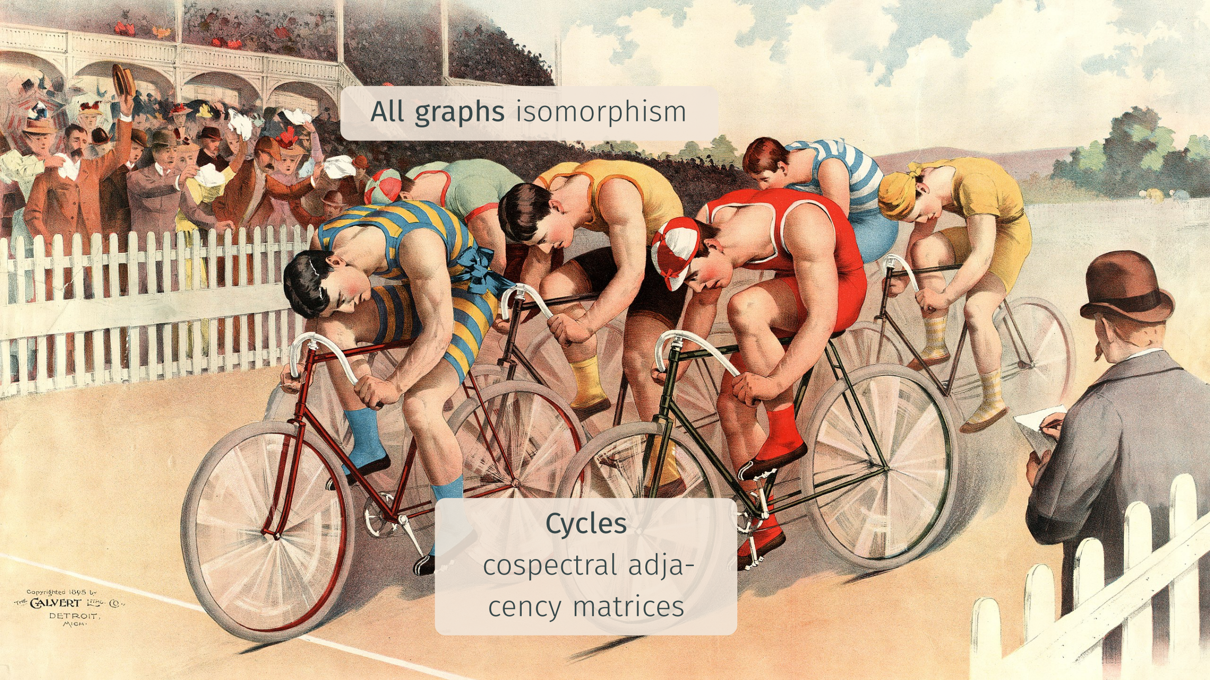
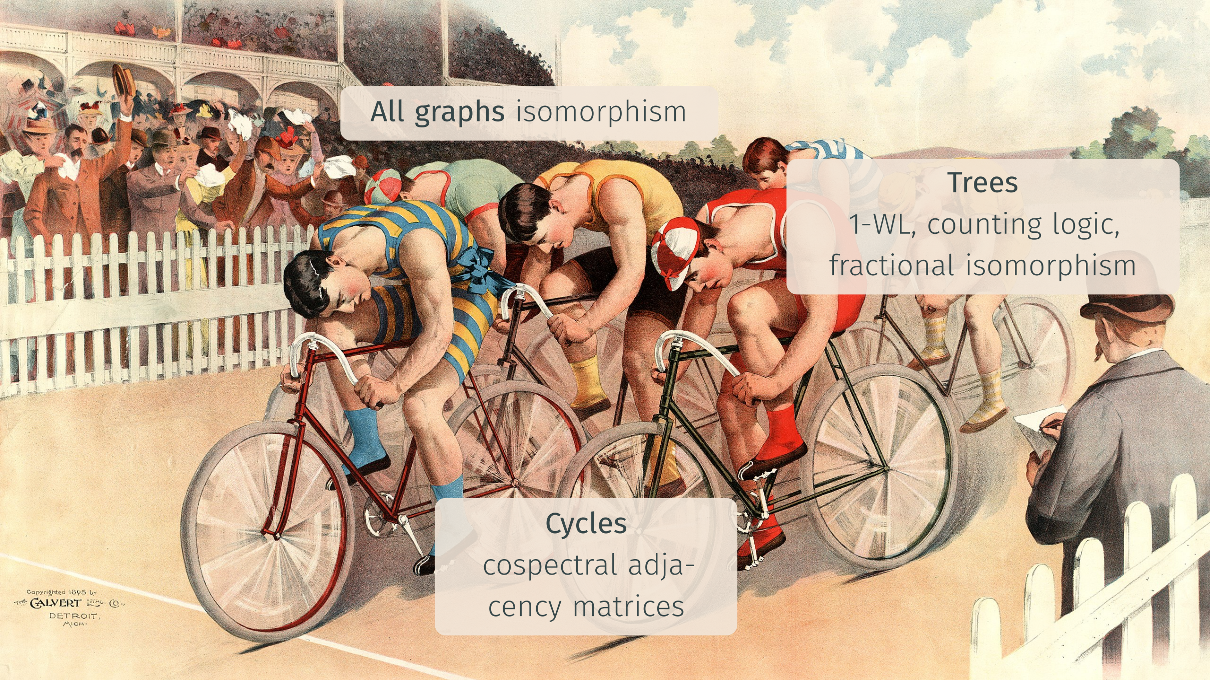
All graphs isomorphism



All graphs isomorphism

Cycles
cospectral adja-
cency matrices

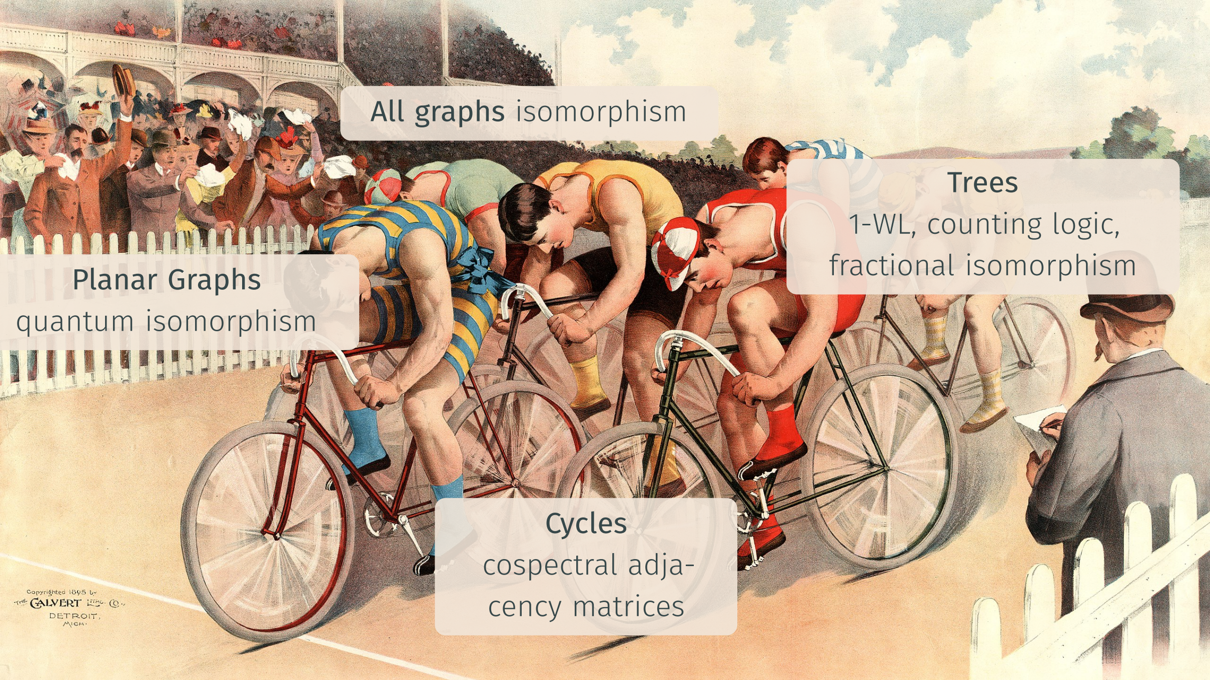




All graphs isomorphism

Trees
1-WL, counting logic, fractional isomorphism

Cycles
cospectral adjacency matrices


All graphs isomorphism
Trees
1-WL, counting logic, fractional isomorphism
Planar Graphs
quantum isomorphism
Cycles
cospectral adja-cency matrices
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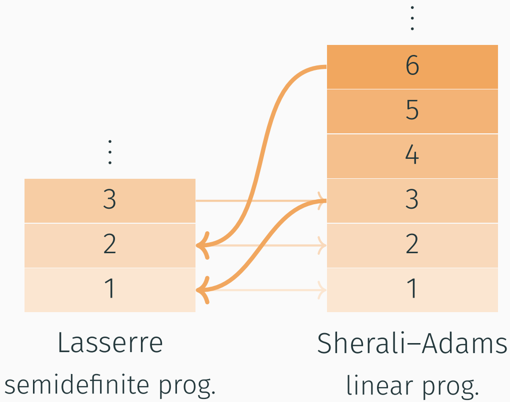

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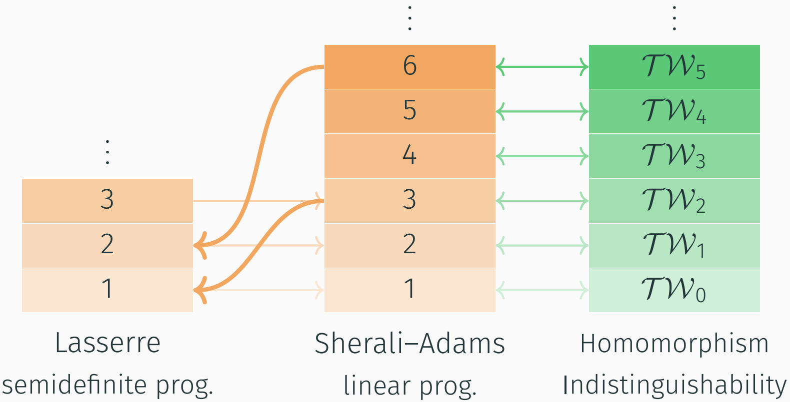

Grohe and Otto (2015), Atserias and Maneva (2012), Dvořák (2010)

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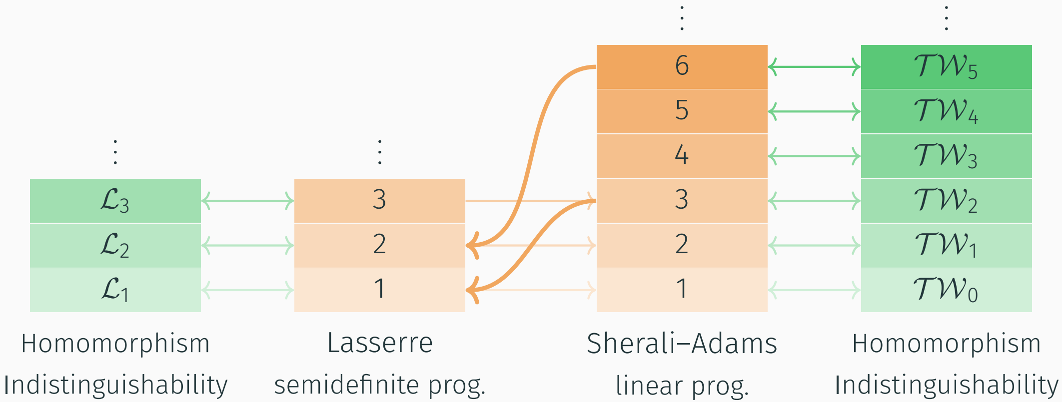

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Equations
homomorphism tensors,
algebraic operations

Graph Class
(bi)labelled graphs,
combinatorial operations


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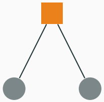
Labelled Graphs and Homomorphism Vectors



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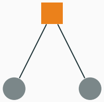

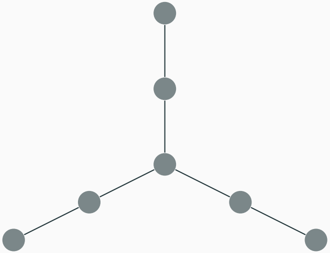

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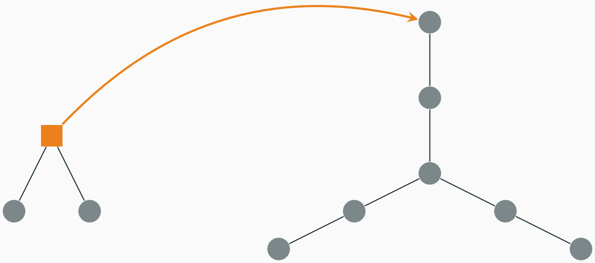

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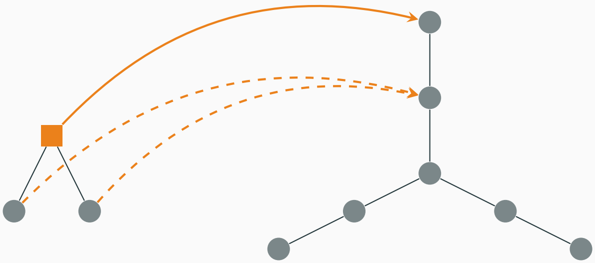

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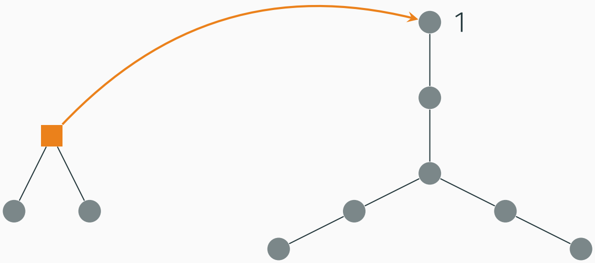

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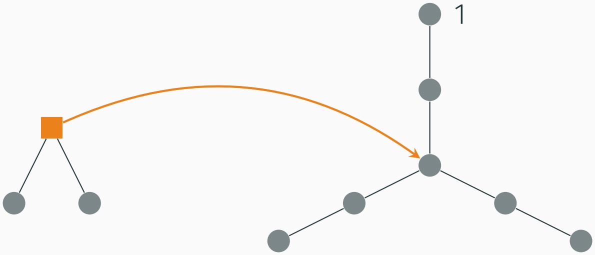

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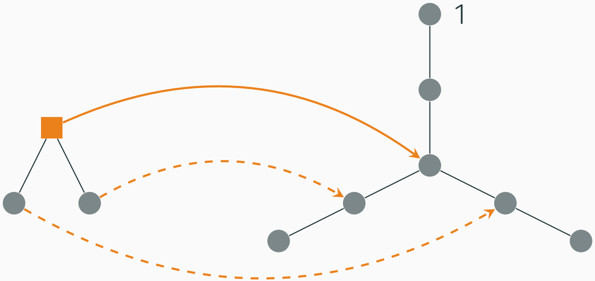
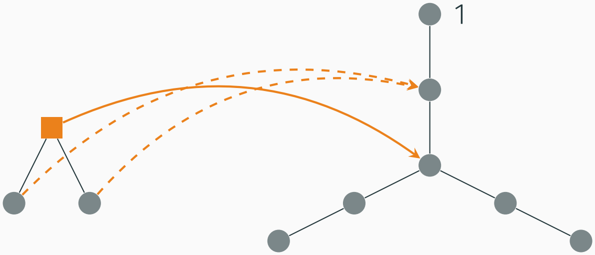
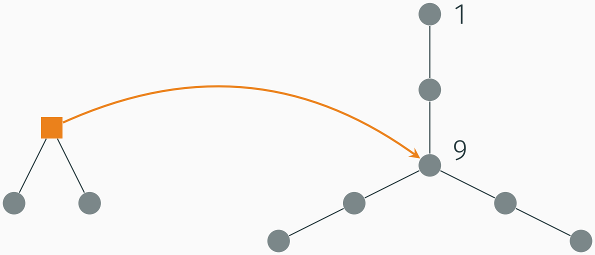
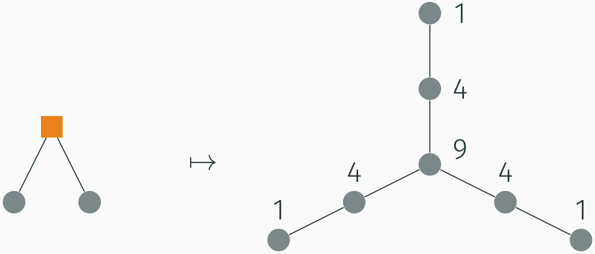



1

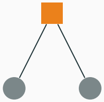
Labelled Graphs and Homomorphism Vectors



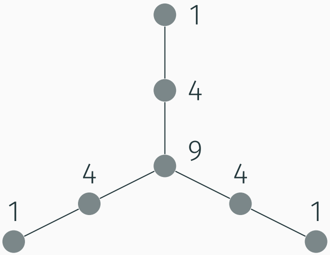
$$\mathcal{F} \longrightarrow \mathbb{C}^{V(G)}$$



Combinatorial and Algebraic Operations: Unlabelling and Sum-of-Entries



$\mapsto$



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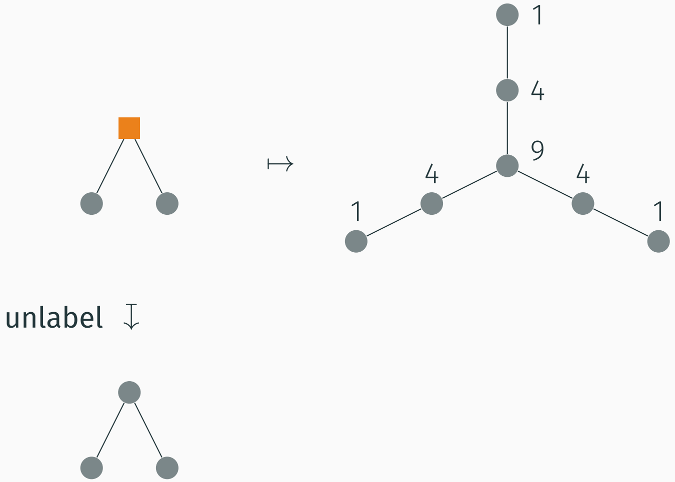

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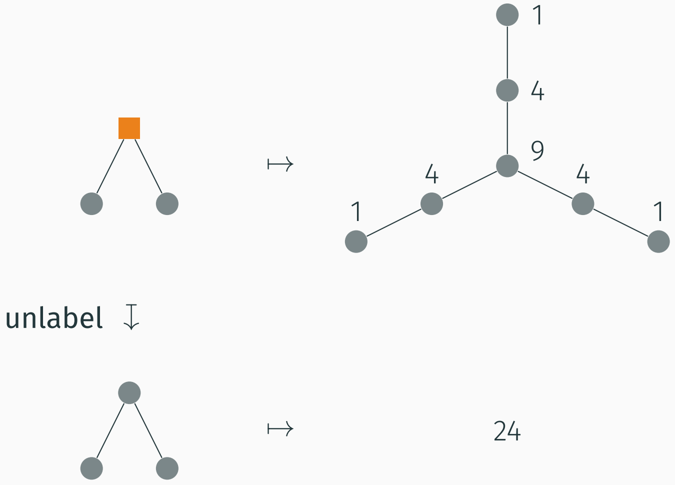
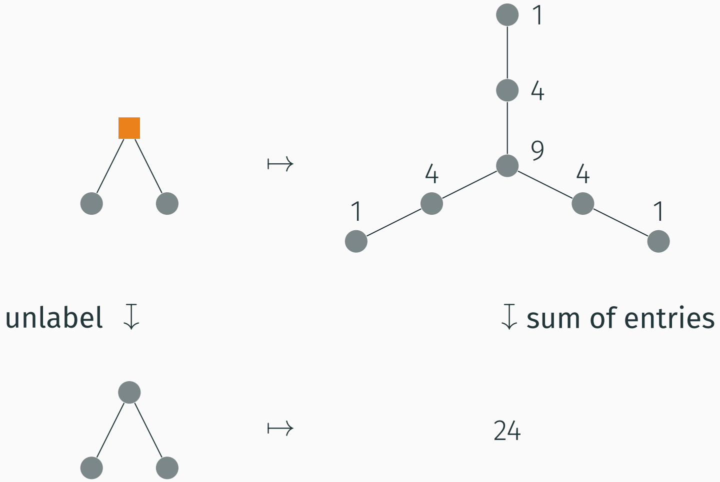
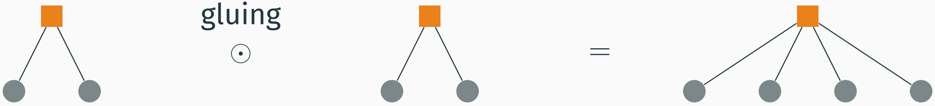



1
4
9
4
4
1
1
↦
unlabel ↧
↧ sum of entries
↦
24

Combinatorial and Algebraic Operations: Gluing and Schur Product



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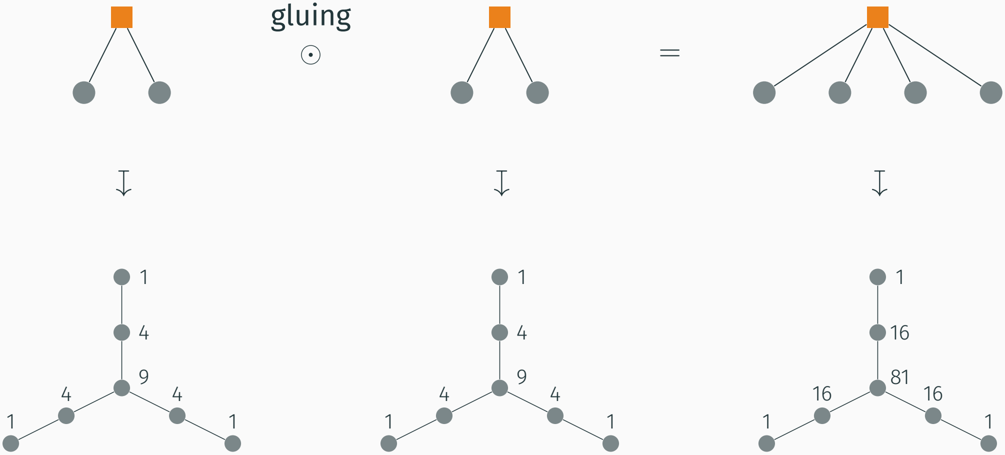

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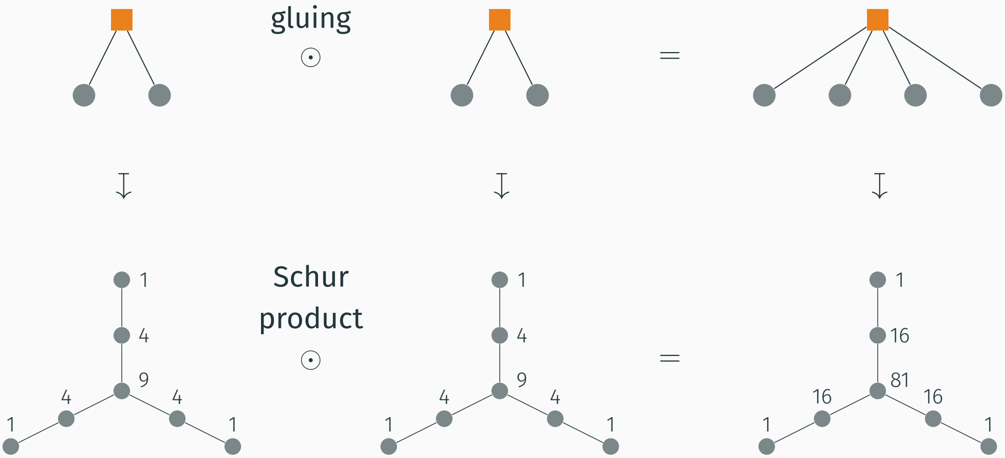
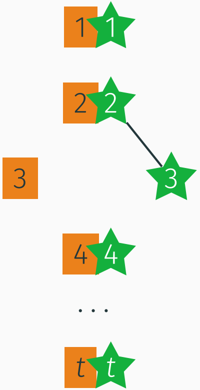


The Graph Class $\mathcal{L}_t$

A (t, t)-bilabelled graph is *atomic* if all its vertices are labelled.



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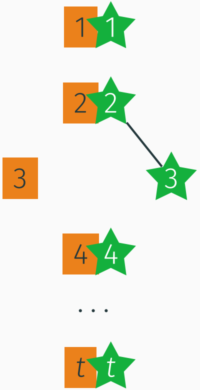
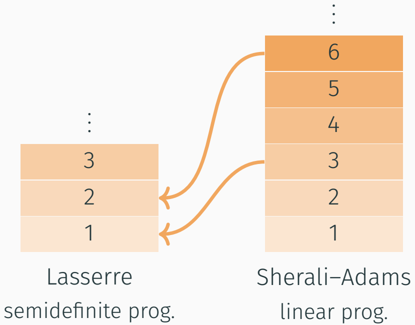
The class $\mathcal{L}_t$ is generated by atomic graphs under

- series composition,
- parallel composition with atomic graphs,
- permutation of labels.



Upper Bound

- $\mathcal{L}_t \subseteq \mathcal{TW}_{3t-1}$,



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- $\mathcal{L}_t \subseteq \mathcal{TW}_{3t-1}$,
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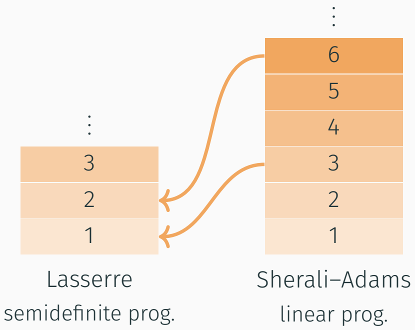
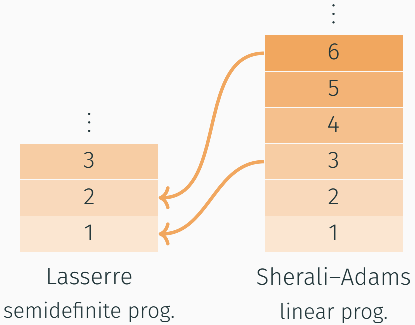


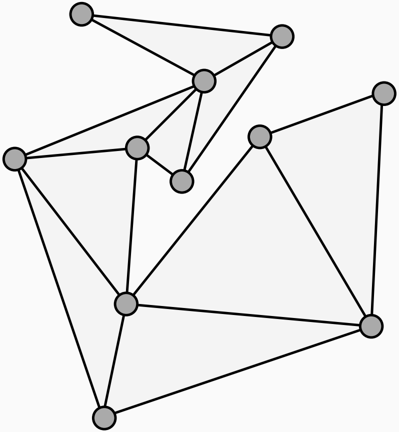
Upper Bound

- $\mathcal{L}_t \subseteq \mathcal{TW}_{3t-1}$,
- $\mathcal{L}_t$ contains the clique K_{3t},
- $\mathcal{L}_t$ is minor-closed,



Upper Bound

- $\mathcal{L}_t \subseteq \mathcal{TW}_{3t-1}$,
- $\mathcal{L}_t$ contains the clique K_{3t},
- $\mathcal{L}_t$ is minor-closed,
- $\mathcal{L}_1$ is the class of all outerplanar graphs.



Lower Bound

$\mathcal{L}_t$ is a class of graphs of treewidth $\leq 3t - 1$ containing K_{3t}.

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Although $\mathcal{L}_t \not\subseteq \mathcal{TW}_{3t-2}$, it could well be that $G \equiv_{\mathcal{TW}_{3t-2}} H \implies G \equiv_{\mathcal{L}_t} H$.

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The homomorphism distinguishing closure of a graph class $\mathcal{F}$ is

$$\mathrm{cl}(\mathcal{F}) = \{K \text{ graph} \mid \forall \text{graphs } G, H.\ G \equiv_{\mathcal{F}} H \implies \hom(K, G) = \hom(K, H)\}.$$

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Conjecture (Roberson (2022))

Every minor-closed union-closed graph class is homomorphism distinguishing closed.

Roberson's Conjecture: State of Affairs

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Theorem (Neuen (2023))

$\mathcal{TW}_k$ is *homomorphism distinguishing closed.*

Roberson's Conjecture: State of Affairs

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Every minor-closed union-closed graph class is homomorphism distinguishing closed.

Theorem (Neuen (2023))

$\mathcal{TW}_k$ is *homomorphism distinguishing closed.*

Corollary

For every $t \geq 1$, there are graphs G and H such that $G \simeq^{\mathsf{SA}}_{3t-1} H$ and $G \not\simeq^{\mathsf{L}}_{t} H$.

G and H are isomorphic iff
integer program $\mathrm{ISO}(G, H)$ is feasible

$\vdots$

3
2
1

Lasserre
semidefinite prog.

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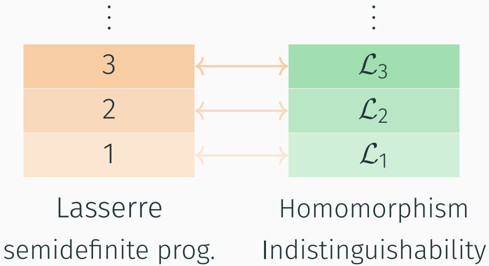

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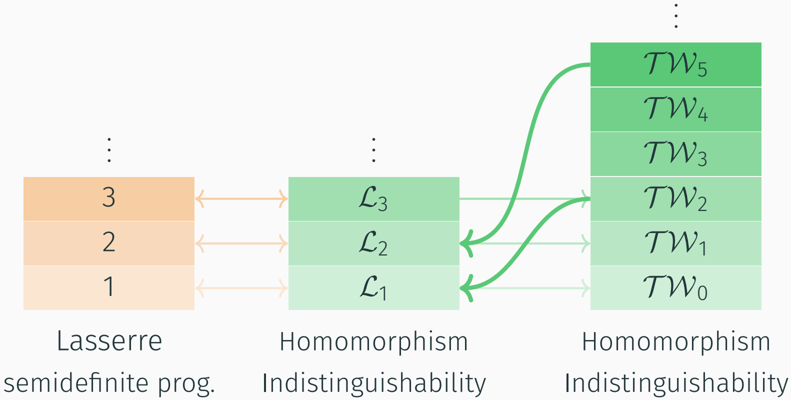

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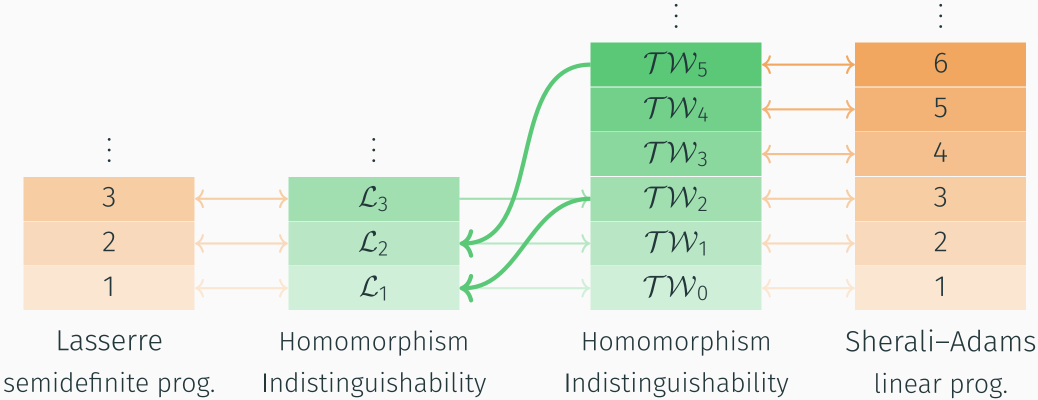
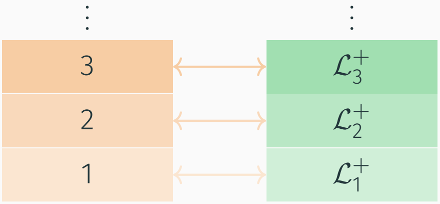


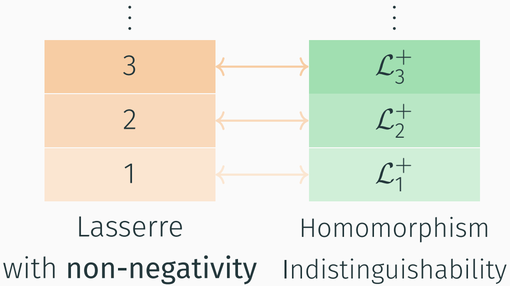
Lasserre with non-negativity constraints



Lasserre
with **non-negativity**

Homomorphism
Indistinguishability

Lasserre with non-negativity constraints

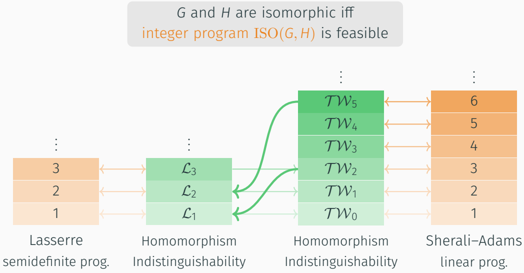


Theorem

Exact feasibility of the level-t Lasserre relaxation with non-negativity constraints of $\mathrm{ISO}(G, H)$ can be decided in polynomial time.

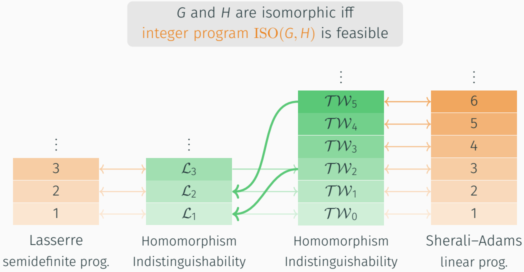
Conclusion

- Determined number of Sherali–Adams levels necessary to guarantee feasibility of Lasserre



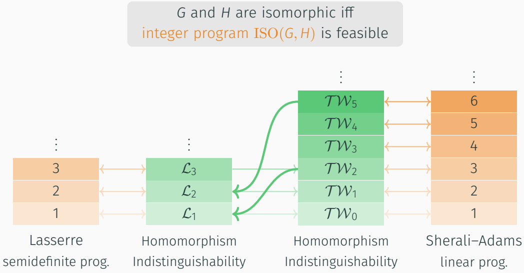
Conclusion

- Determined number of Sherali–Adams levels necessary to guarantee feasibility of Lasserre
- Homomorphism indistinguishability characterisations



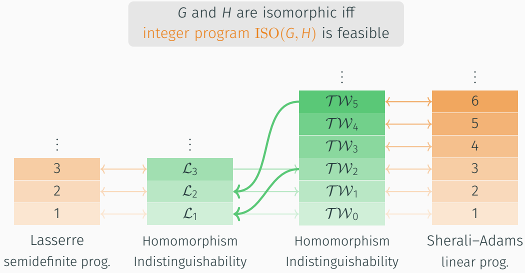
Conclusion

- Determined number of Sherali–Adams levels necessary to guarantee feasibility of Lasserre
- Homomorphism indistinguishability characterisations
- PTIME algorithm for non-negative Lasserre



Conclusion

- Determined number of Sherali–Adams levels necessary to guarantee feasibility of Lasserre
- Homomorphism indistinguishability characterisations
- PTIME algorithm for non-negative Lasserre
- What about the number of Lasserre levels necessary to guarantee feasibility of Sherali–Adams?



Extras: Lasserre

Let $t \geq 1$. The *level-t Lasserre relaxation for graph isomorphism* has variables y_I ranging over $\mathbb{R}$ for $I \in \binom{V(G)\times V(H)}{\leq 2t}$. The constraints are

$$
\begin{aligned}
M_t(y) := (y_{I\cup J})_{I,J\in\binom{V(G)\times V(H)}{\leq t}} &\succeq 0, \\
\sum_{h\in V(H)} y_{I\cup\{gh\}} &= y_I \text{ for all } I \text{ s.t. } |I| \leq 2t-2 \text{ and all } g \in V(G), \\
\sum_{g\in V(G)} y_{I\cup\{gh\}} &= y_I \text{ for all } I \text{ s.t. } |I| \leq 2t-2 \text{ and all } h \in V(H), \\
y_I &= 0 \text{ if } I \text{ s.t. } |I| \leq 2t \text{ is not partial isomorphism} \\
y_\emptyset &= 1.
\end{aligned}
$$

Extras: Sherali–Adams

Let $t \geq 1$. The *level-t Sherali–Adams relaxation for graph isomorphism* has variables y_I ranging over $\mathbb{R}$ for $I \in \binom{V(G)\times V(H)}{\leq t}$. The constraints are

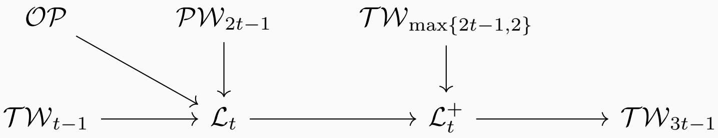
$$\sum_{h \in V(H)} y_{I \cup \{gh\}} = y_I \text{ for all } I \text{ s.t. } |I| \leq t-1 \text{ and all } g \in V(G),$$

$$\sum_{g \in V(G)} y_{I \cup \{gh\}} = y_I \text{ for all } I \text{ s.t. } |I| \leq t-1 \text{ and all } h \in V(H),$$

$$y_I = 0 \text{ if } I \text{ s.t. } |I| \leq t \text{ is not partial isomorphism}$$

$$y_\emptyset = 1.$$

Extra: Graph Classes



References

Atserias, A. and Maneva, E. (2012). Sherali–Adams Relaxations and Indistinguishability in Counting Logics. In *Proceedings of the 3rd Innovations in Theoretical Computer Science Conference*, ITCS '12, pages 367–379, New York, NY, USA. Association for Computing Machinery.

Atserias, A. and Maneva, E. (2013). Sherali–Adams Relaxations and Indistinguishability in Counting Logics. *SIAM Journal on Computing*, 42(1):112–137.

Atserias, A. and Ochremiak, J. (2018). Definable ellipsoid method, sums-of-squares proofs, and the isomorphism problem. In Dawar, A. and Grädel, E., editors, *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018, Oxford, UK, July 09-12, 2018*, pages 66–75. ACM.

Dvořák, Z. (2010). On recognizing graphs by numbers of homomorphisms. *Journal of Graph Theory*, 64(4):330–342.

Grohe, M. and Otto, M. (2015). Pebble Games and Linear Equations. *The Journal of Symbolic Logic*, 80(3):797–844.

Neuen, D. (2023). Homomorphism-Distinguishing Closedness for Graphs of Bounded Tree-Width. arXiv:2304.07011 [cs, math].

Roberson, D. E. (2022). Oddomorphisms and homomorphism indistinguishability over graphs of bounded degree. Number: arXiv:2206.10321.

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