
# Syntax and Semantics of Homomorphism Indistinguishability

Coresources 2023

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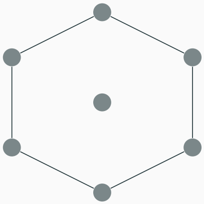
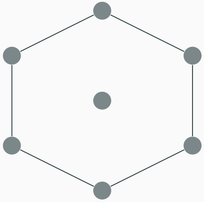
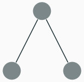
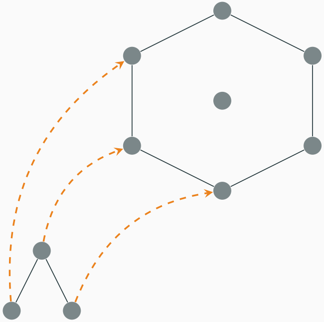
UnRAVeL
RWTH AACHEN UNIVERSITY

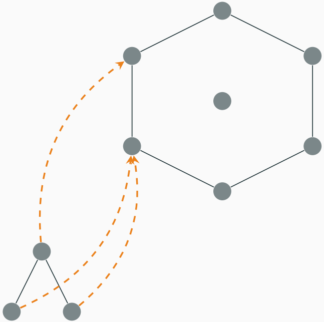
DFG Deutsche Forschungsgemeinschaft
German Research Foundation

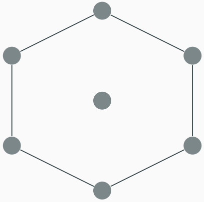


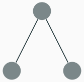












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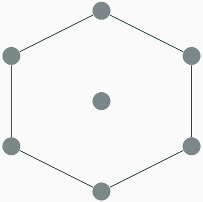

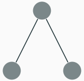

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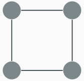

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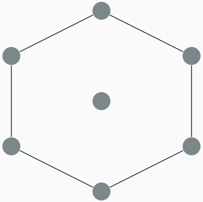
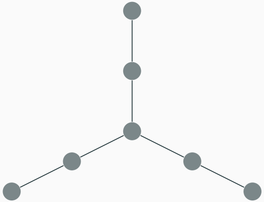
| |  |  |
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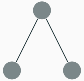
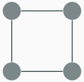
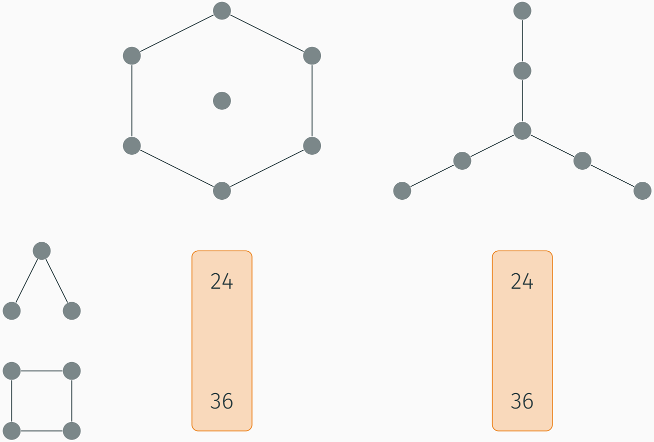
|  | 24 | 24 |
|  | 36 | 36 |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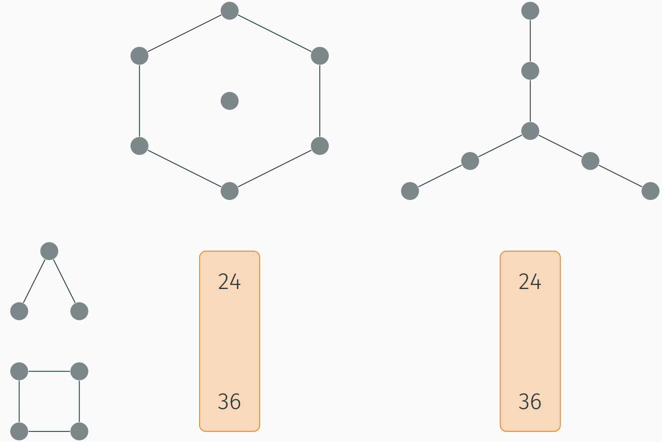
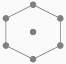
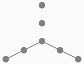



The graphs  and  are **homomorphism indistinguishable** over {, }.

| **graph class $\mathcal{F}$** | **relation $\equiv_{\mathcal{F}}$** | |
|---|---|---|
| all graphs | isomorphism | Lovász (1967) |
| planar graphs | quantum isomorphism | Mančinska and Roberson (2020) |
| treewidth $\leq k$ | $\mathsf{C}^{k+1}$-equivalence | Dvořák (2010) |

# Homomorphism Indistinguishability

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| $U^{\mathfrak{C}}\,\mathrm{EM}_f(\mathfrak{C})$ | $\cong_{\mathcal{K}(\mathfrak{C})}$ of finite-rank comonad $\mathfrak{C}$ | Reggio (2021) |
| ... | ... | |

Measuring the power of homomorphism indistinguishability relations

Characterising homomorphism indistinguishability relations

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$G$ and $H$ are isomorphic iff
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| Lasserre | Sherali–Adams |
|---|---|
| ⋮ | ⋮ |
| 3 | 3 |
| 2 | 2 |
| 1 | 1 |
| Lasserre | Sherali–Adams |
| semidefinite prog. | linear prog. |

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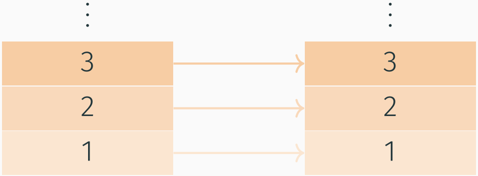


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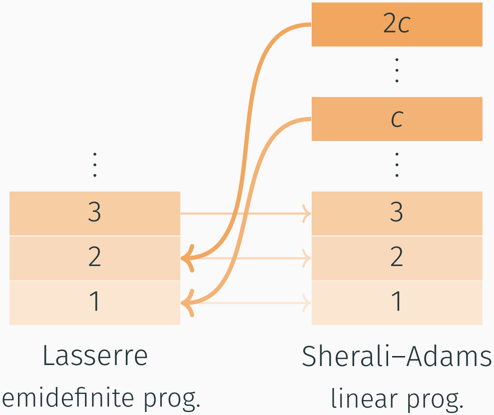


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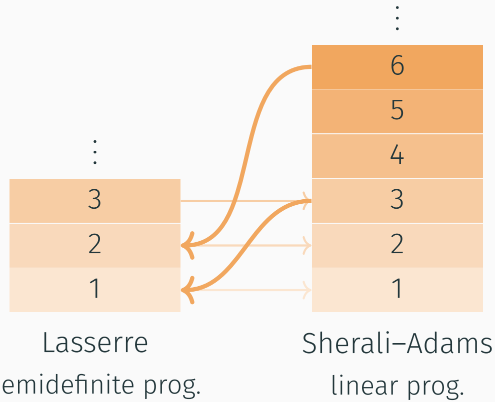


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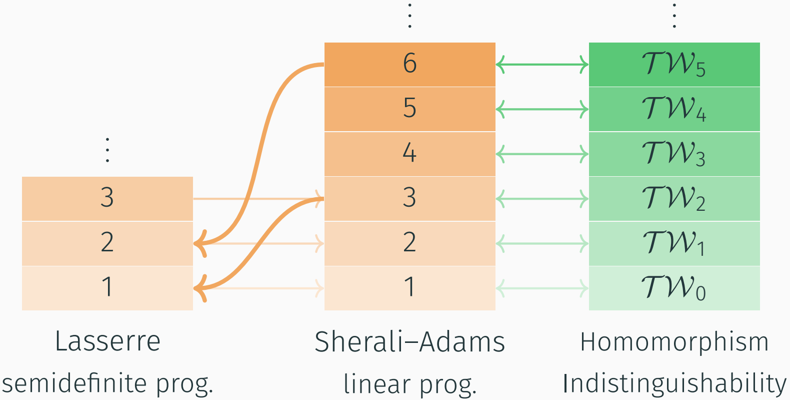


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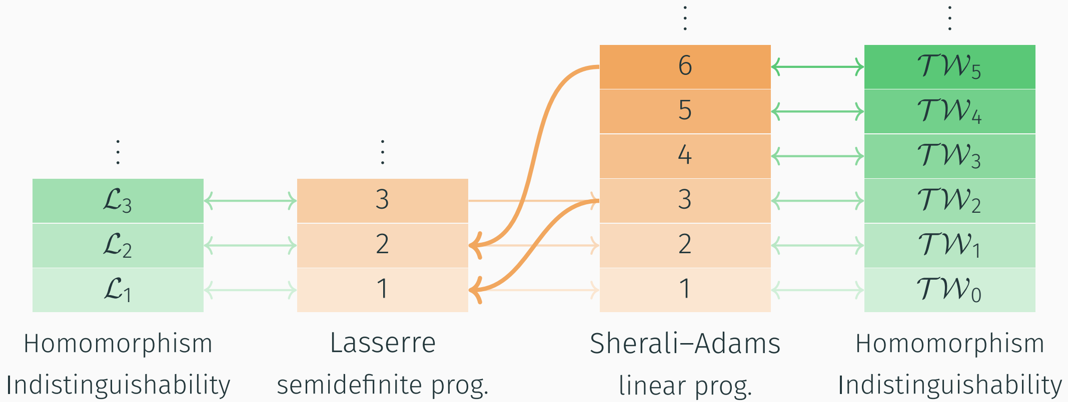


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homomorphism tensors,
algebraic operations

**Graph Class**
(bi)labelled graphs,
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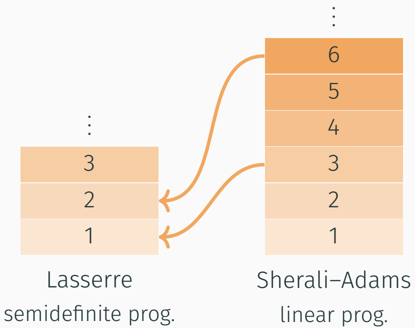
# Upper Bound

- $\mathcal{L}_t \subseteq \mathcal{TW}_{3t-1}$,



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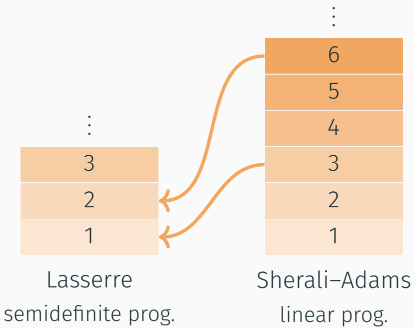

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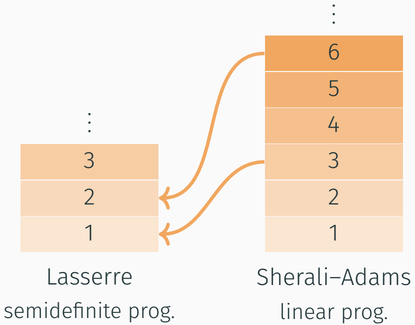

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**Corollary (Roberson and S. (2023))**

*For every $t \geq 1$, there are graphs $G$ and $H$ such that $G \simeq^{SA}_{3t-1} H$ and $G \not\equiv^{L}_{t} H$.*

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⋮

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Lasserre
semidefinite prog.

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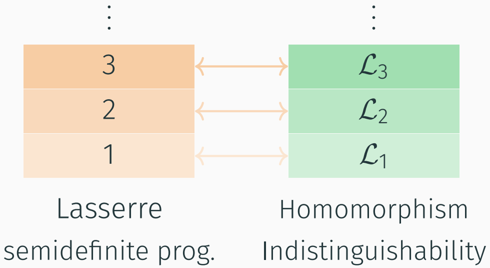

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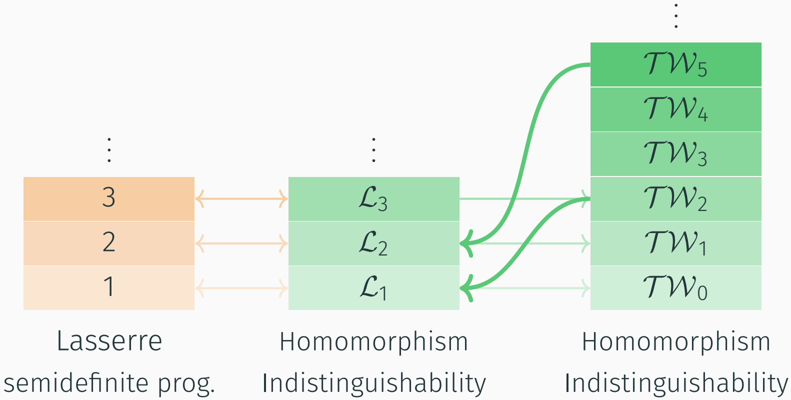

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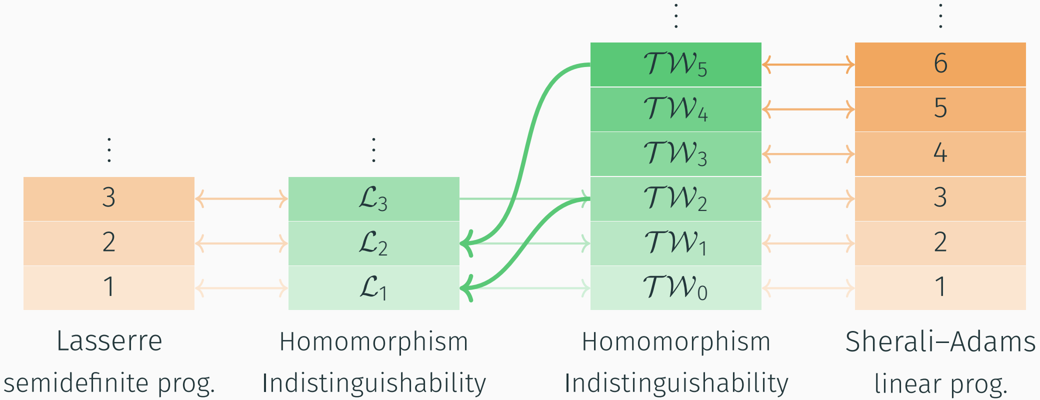

# Characterising homomorphism indistinguishability relations

# Properties of Homomorphism Indistinguishability Relations

**Observation ($\equiv_{\mathcal{F}}$ is preserved under categorical products)**

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For every graph $F$,

$$\hom(F, G_1 \times G_2) = \hom(F, G_1) \hom(F, G_2).$$

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| *subgraphs* | *full complements* | $G \mapsto \widehat{G}$ |
| *induced subgraphs* | *left lexicographic products* | $H \mapsto G[H]$ *for every* $G$ |
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# Self-complementary Logics

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- ***Self-complementary logics*** $(\mathsf{L}, \models)$
  For every sentence $\varphi \in \mathsf{L}$, there is $\overline{\varphi} \in \mathsf{L}$ such that $G \models \varphi \iff \overline{G} \models \overline{\varphi}$.
  E.g., replace $Exy$ by $\neg Exy \wedge (x \neq y)$.

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$K_k \equiv_{\mathsf{FO}^k} K_{k+1}$ but $\hom(K_1, K_k) \neq \hom(K_1, K_{k+1})$, so $K_1 \notin \mathcal{F}$, contradiction!

# Graph Minor Theory rules out Homomorphism Indistinguishability

**Theorem (Robertson and Seymour (1986))**

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*Then all $\mathsf{L}$-equivalent graphs are quantum isomorphic.*

# Outlook: Logics stronger than Weisfeiler–Leman

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**Corollary**

$\mathsf{LA}^k(Q)$*-equivalence cannot be characterised as co-Kleisli isomorphism w.r.t. any comonad of finite rank.*

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**Input** graphs $G$ and $H$

**Decide** $G \equiv_{\mathcal{F}} H$.

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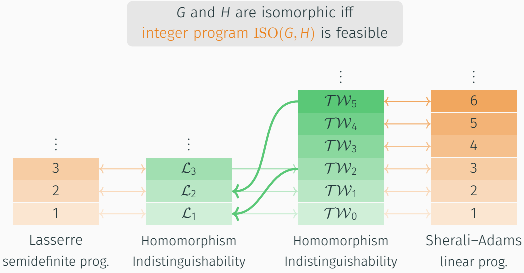
**Question**

*For a minor-closed $\mathcal{F}$, either*

| | | |
|---|---|---|
| *$\mathcal{F}$ contains all graphs* | *and* | *$\textsc{HomInd}(\mathcal{F})$ is Graph Isomorphism,* |
| *$\mathcal{F}$ has bounded treewidth* | *and* | *$\textsc{HomInd}(\mathcal{F})$ is decidable, or* |
| *$\mathcal{F}$ has unbounded treewidth* | *and* | *$\textsc{HomInd}(\mathcal{F})$ is undecidable.* |

# Conclusion

- Roberson's conjecture and homomorphism distinguishing closure



# Conclusion

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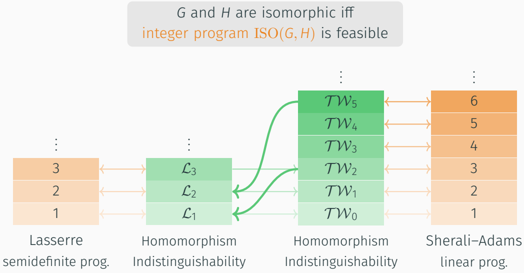
- Closure properties correspond to preservation properties



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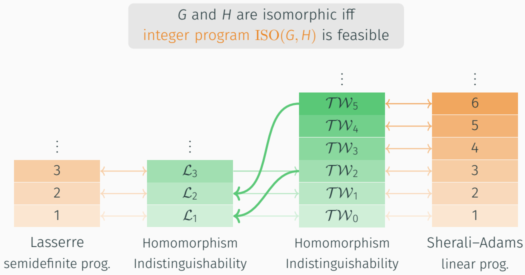

# Conclusion

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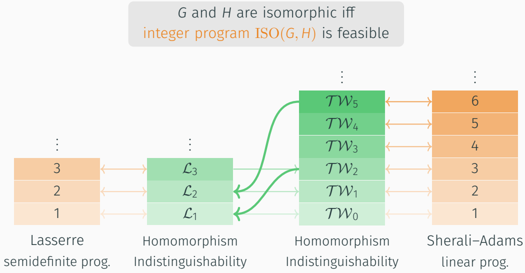
- Check out 2302.11290 and 2302.10538!



# Extras: Lasserre

Let $t \geq 1$. The *level-t Lasserre relaxation for graph isomorphism* has variables $y_I$ ranging over $\mathbb{R}$ for $I \in \binom{V(G)\times V(H)}{\leq 2t}$. The constraints are

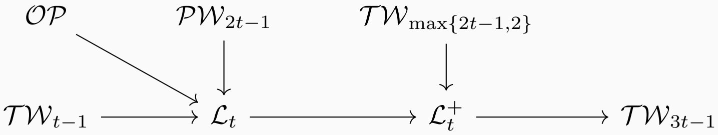
$$\begin{aligned}
M_t(y) := (y_{I\cup J})_{I,J\in\binom{V(G)\times V(H)}{\leq t}} &\succeq 0,\\
\sum_{h\in V(H)} y_{I\cup\{gh\}} &= y_I \text{ for all } I \text{ s.t. } |I| \leq 2t-2 \text{ and all } g \in V(G),\\
\sum_{g\in V(G)} y_{I\cup\{gh\}} &= y_I \text{ for all } I \text{ s.t. } |I| \leq 2t-2 \text{ and all } h \in V(H),\\
y_I &= 0 \text{ if } I \text{ s.t. } |I| \leq 2t \text{ is not partial isomorphism}\\
y_\emptyset &= 1.
\end{aligned}$$

# Extras: Sherali–Adams

Let $t \geq 1$. The *level-$t$ Sherali–Adams relaxation for graph isomorphism* has variables $y_I$ ranging over $\mathbb{R}$ for $I \in \binom{V(G)\times V(H)}{\leq t}$. The constraints are

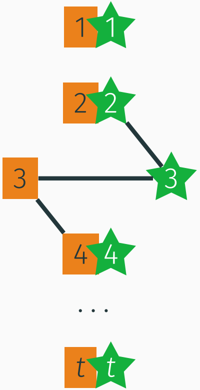
$$\sum_{h\in V(H)} y_{I\cup\{gh\}} = y_I \text{ for all } I \text{ s.t. } |I| \leq t-1 \text{ and all } g \in V(G),$$
$$\sum_{g\in V(G)} y_{I\cup\{gh\}} = y_I \text{ for all } I \text{ s.t. } |I| \leq t-1 \text{ and all } h \in V(H),$$
$$y_I = 0 \text{ if } I \text{ s.t. } |I| \leq t \text{ is not partial isomorphism}$$
$$y_\emptyset = 1.$$

# Extra: Graph Classes



# The Graph Class $\mathcal{L}_t$

A $(t,t)$-bilabelled graph is *atomic* if all its vertices are labelled.

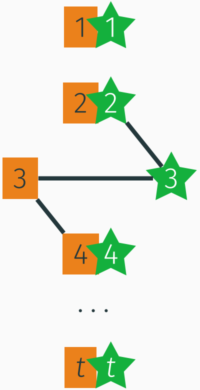


# The Graph Class $\mathcal{L}_t$

A $(t, t)$-bilabelled graph is *atomic* if all its vertices are labelled.

The class $\mathcal{L}_t$ is generated by atomic graphs under

- series composition,
- parallel composition with atomic graphs,
- permutation of labels.



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