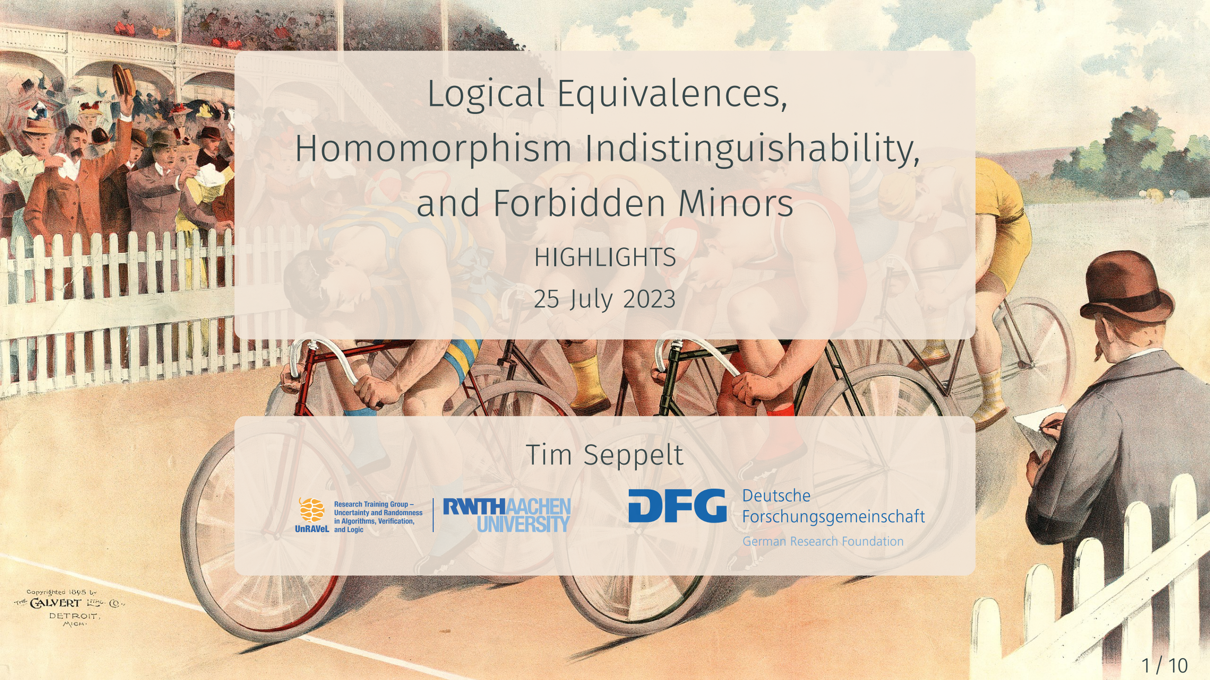


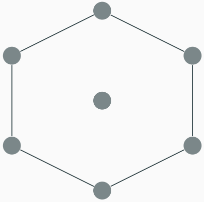
Logical Equivalences, Homomorphism Indistinguishability, and Forbidden Minors

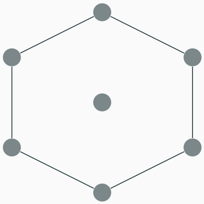
HIGHLIGHTS

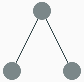
25 July 2023

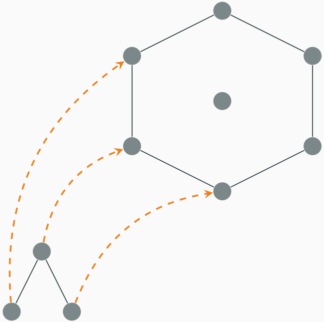
Tim Seppelt

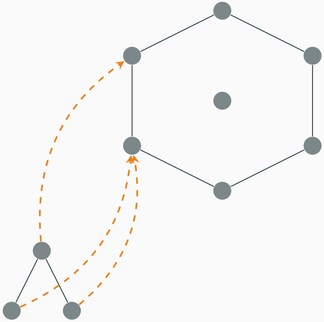
Research Training Group – Uncertainty and Randomness in Algorithms, Verification, and Logic
UnRAVeL
RWTH Aachen University
DFG Deutsche Forschungsgemeinschaft
German Research Foundation

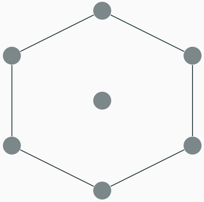


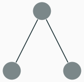












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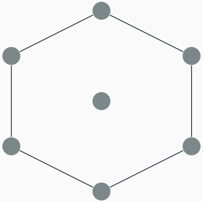

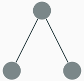

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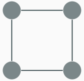

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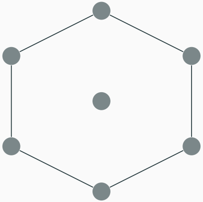
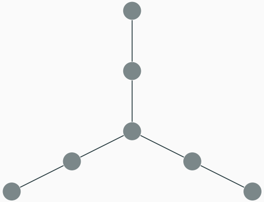
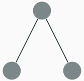
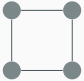
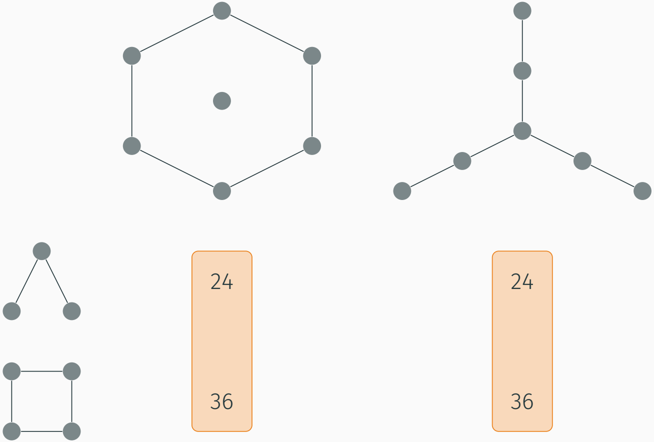
		
	24	24
	36	36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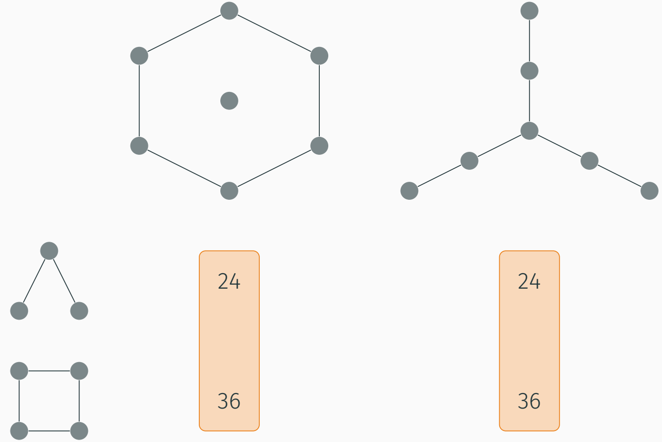
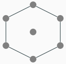
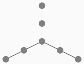


The graphs  and  are **homomorphism indistinguishable** over $\{$, $\}$.

Homomorphism Indistinguishability

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treewidth $\leq k$	C^{k+1}-equivalence	Dvořák (2010)
treedepth $\leq d$	C_d-equivalence	Grohe (2020)
...	...	

When is an equivalence relation between graphs a homomorphism indistinguishability relation?

Observation ($\equiv_{\mathcal{F}}$ is preserved under categorical products)

If $G_1 \equiv_{\mathcal{F}} H_1$ and $G_2 \equiv_{\mathcal{F}} H_2$ then $G_1 \times G_2 \equiv_{\mathcal{F}} H_1 \times H_2$.

Properties of Homomorphism Indistinguishability Relations

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For every graph F,

$$\hom(F, G_1 \times G_2) = \hom(F, G_1) \hom(F, G_2).$$

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subgraphs	*full complements*	$G \mapsto \widehat{G}$
induced subgraphs	*left lexicographic products*	$H \mapsto G[H]$ *for every* G
contracting edges	*right lexicographic products*	$G \mapsto G[H]$ *for every* H.

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- **Feasibility of integer programming relaxations for graph isomorphism**
  Graphs are encoded via atomic types of vertex tuples
- ***Self-complementary logics*** $(\mathsf{L}, \models)$
  For every sentence $\varphi \in \mathsf{L}$, there is $\overline{\varphi} \in \mathsf{L}$ such that $G \models \varphi \iff \overline{G} \models \overline{\varphi}$.
  E.g., replace Exy by $\neg Exy \wedge (x \neq y)$.

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$K_k \equiv_{\mathsf{FO}^k} K_{k+1}$ but $\hom(K_1, K_k) \neq \hom(K_1, K_{k+1})$, so $K_1 \notin \mathcal{F}$, contradiction!

Graph Minor Theory rules out Homomorphism Indistinguishability

Theorem (Robertson and Seymour (1986))

For a minor-closed graph class $\mathcal{F}$, tfae:

- *$\mathcal{F}$ has unbounded treewidth,*
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Then all L-equivalent graphs are quantum isomorphic.

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- Check out `arXiv:2302.11290`!

References

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