Logical Equivalences, Homomorphism Indistinguishability, and Forbidden Minors

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DFG Deutsche Forschungsgemeinschaft
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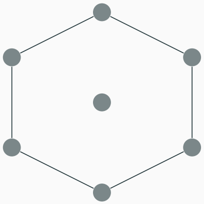

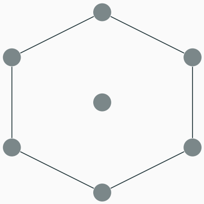

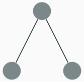

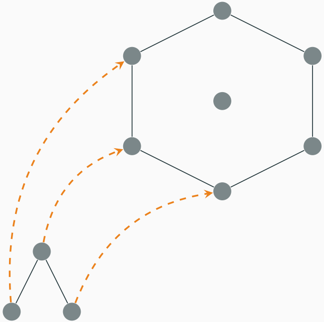

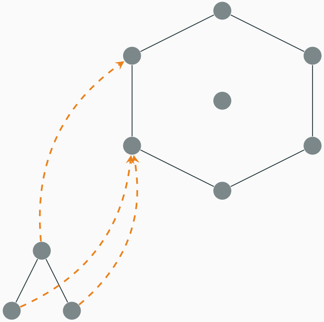

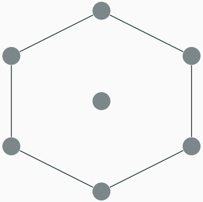

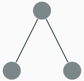

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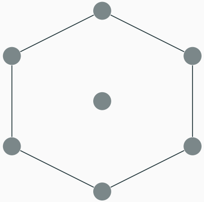

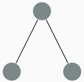

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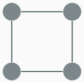

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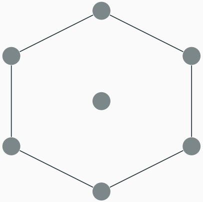
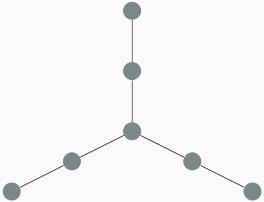
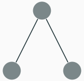
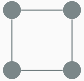
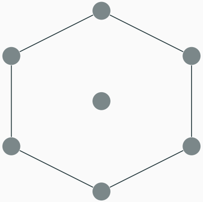
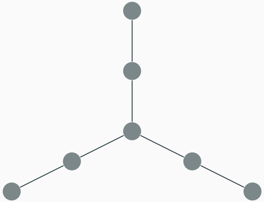
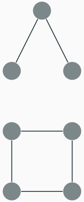
		
	24	24
	36	36







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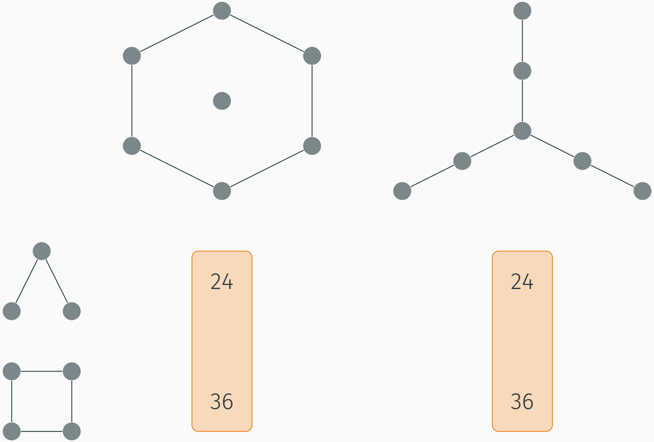
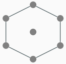
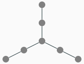


The graphs  and  are **homomorphism indistinguishable** over $\{$, $\}$.

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graph class $\mathcal{F}$	relation $\equiv_{\mathcal{F}}$	
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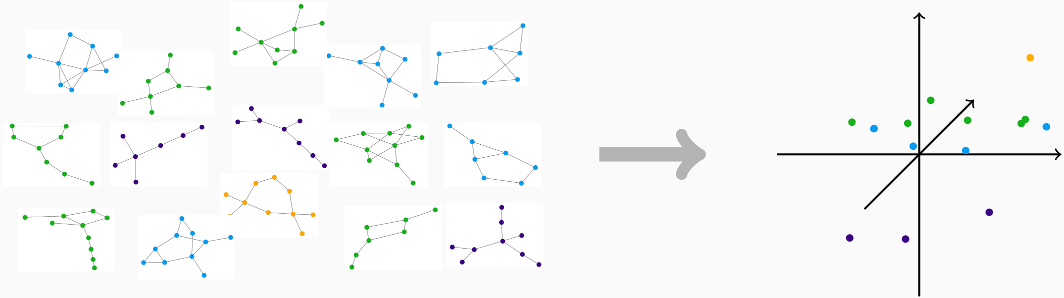
	$\mathbb{P}_k$-coKleisli-isomorphism	Dawar et al. (2021)
...	...	



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- Compare power of relations $\equiv_{\mathcal{F}_1}$ and $\equiv_{\mathcal{F}_2}$ by comparing graph classes $\mathcal{F}_1$ and $\mathcal{F}_2$ Roberson and S. (2023)

When is an equivalence relation between graphs a homomorphism indistinguishability relation?

Observation ($\equiv_{\mathcal{F}}$ is preserved under categorical products)

If $G_1 \equiv_{\mathcal{F}} H_1$ and $G_2 \equiv_{\mathcal{F}} H_2$ then $G_1 \times G_2 \equiv_{\mathcal{F}} H_1 \times H_2$.

Properties of Homomorphism Indistinguishability Relations

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For every graph F,

$$\hom(F, G_1 \times G_2) = \hom(F, G_1)\hom(F, G_2).$$

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summands	*disjoint unions*	$(G, H) \mapsto G + H$
subgraphs	*full complements*	$G \mapsto \widehat{G}$
induced subgraphs	*left lexicographic products*	$H \mapsto G[H]$ *for every* G
contracting edges	*right lexicographic products*	$G \mapsto G[H]$ *for every* H.

Homomorphism Distinguishing Closure

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for all $F \notin \mathcal{F}$ there exist G and H such that $G \equiv_{\mathcal{F}} H$ and $\hom(F, G) \neq \hom(F, H)$.

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- **essentially finite graph classes**, — generalising Kwiecień et al. (2022)

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- **Feasibility of integer programming relaxations for graph isomorphism**
  Graphs are encoded via atomic types of vertex tuples
- ***Self-complementary logics*** $(\mathsf{L}, \models)$
  For every sentence $\varphi \in \mathsf{L}$, there is $\overline{\varphi} \in \mathsf{L}$ such that $G \models \varphi \iff \overline{G} \models \overline{\varphi}$.
  E.g., replace Exy by $\neg Exy \wedge (x \neq y)$.

Ruling out Homomorphism Indistinguishability Relations

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$K_k \equiv_{\mathsf{FO}^k} K_{k+1}$ but $\hom(K_1, K_k) \neq \hom(K_1, K_{k+1})$, so $K_1 \notin \mathcal{F}$, contradiction!

Graph Minor Theory rules out Homomorphism Indistinguishability

Theorem (Robertson and Seymour (1986))

For a minor-closed graph class $\mathcal{F}$, tfae:

- *$\mathcal{F}$ has unbounded treewidth,*
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Then all L-equivalent graphs are quantum isomorphic.

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- When is an equivalence relation between graphs a homomorphism indistinguishability relation?

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Pictures

- "Bicycle race scene. A peloton of six cyclists crosses the finish line in front of a crowded grandstand, observed by a referee." (1895) by Calvert Lithographic Co., Detroit, Michigan, Public Domain, via Wikimedia Commons. `https://commons.wikimedia.org/wiki/File:Bicycle_race_scene,_1895.jpg`
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