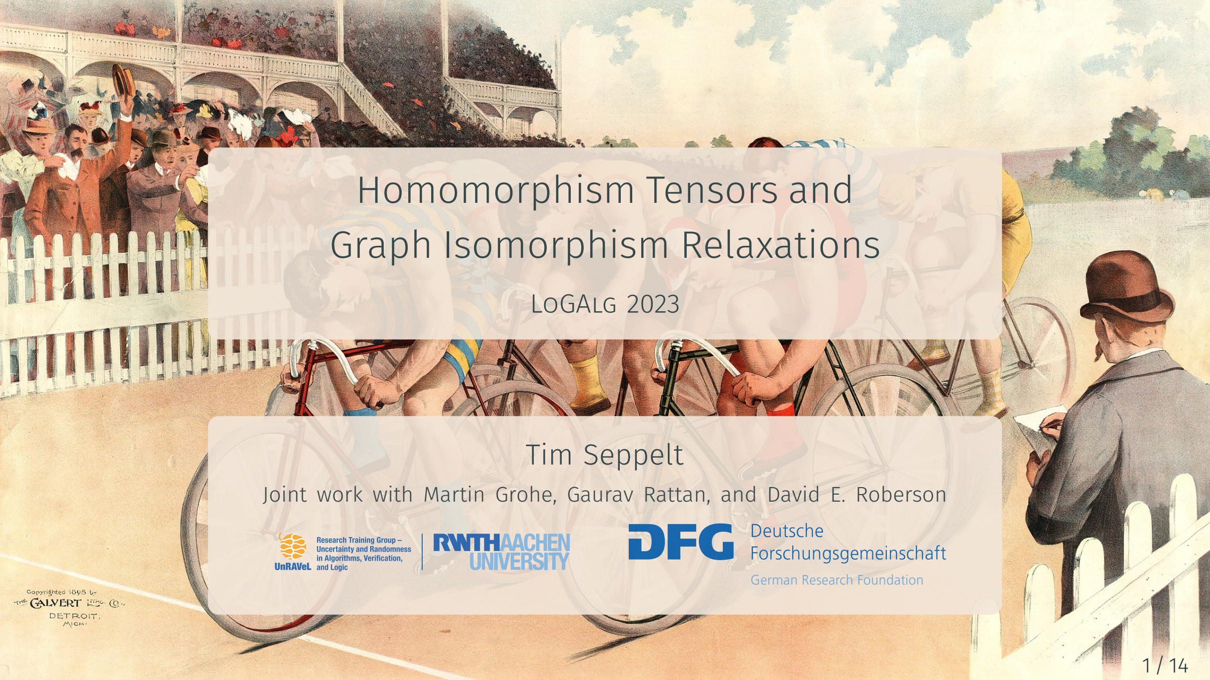

Homomorphism Tensors and Graph Isomorphism Relaxations

LoGAlg 2023

Tim Seppelt

Joint work with Martin Grohe, Gaurav Rattan, and David E. Roberson

UnRAVeL Research Training Group – Uncertainty and Randomness in Algorithms, Verification, and Logic

RWTH AACHEN UNIVERSITY

DFG Deutsche Forschungsgemeinschaft
German Research Foundation

G and H are isomorphic iff
integer program $\mathrm{ISO}(G, H)$ is feasible

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⋮	⋮
3	3
2	2
1	1
Lasserre	Sherali–Adams
semidefinite prog.	linear prog.

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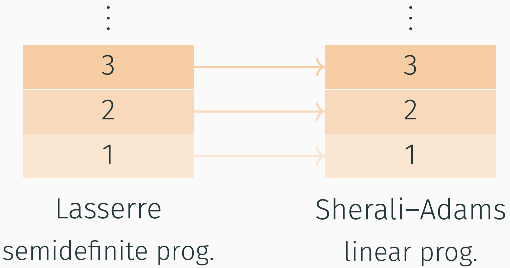

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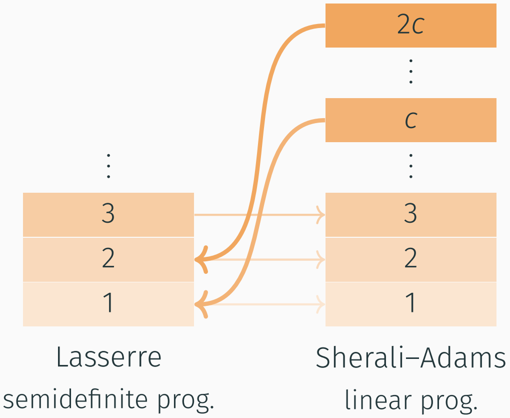

Atserias and Ochremiak (2018)

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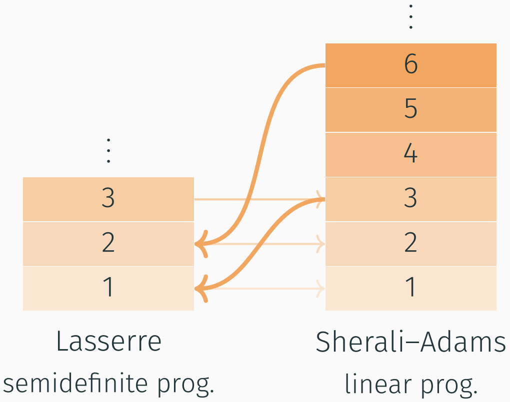
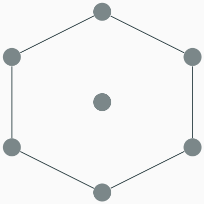
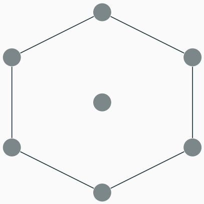
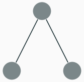
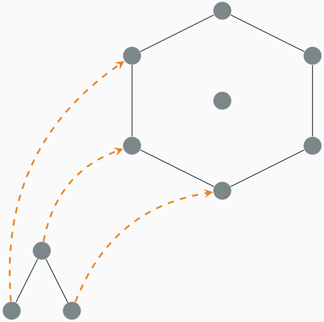


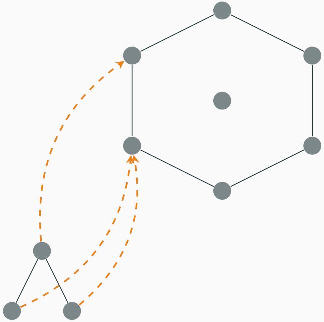
Atserias and Ochremiak (2018), **Roberson & S. (ICALP 2023)**

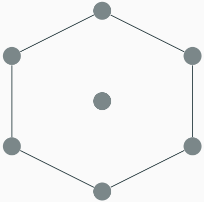


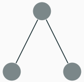












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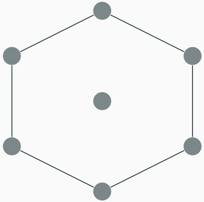

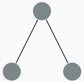

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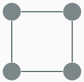

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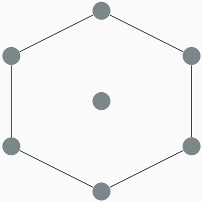
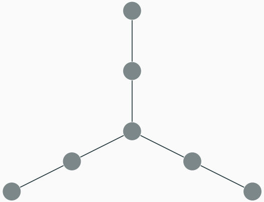
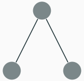
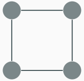
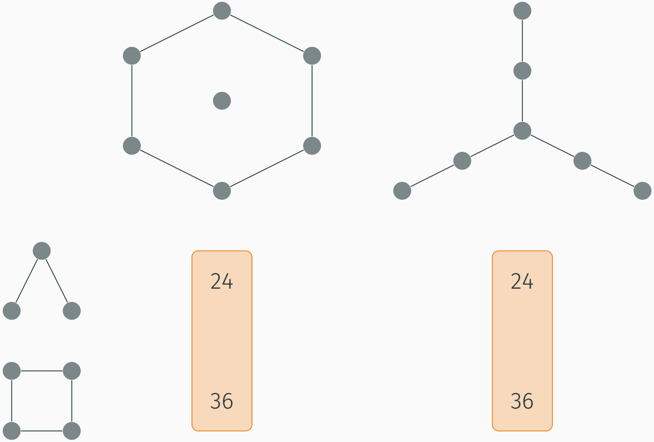
		
	24	24
	36	36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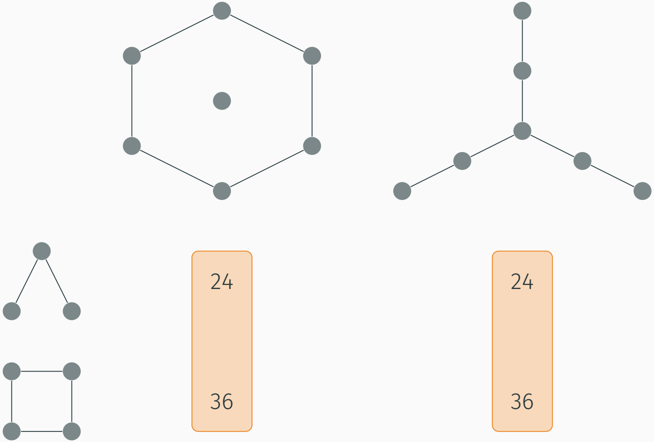
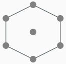
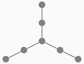
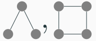


The graphs  and  are **homomorphism indistinguishable** over $\{$$\}$.

Homomorphism Indistinguishability

graph class $\mathcal{F}$	**relation $\equiv_{\mathcal{F}}$**	
all graphs	isomorphism	Lovász (1967)

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...	...	

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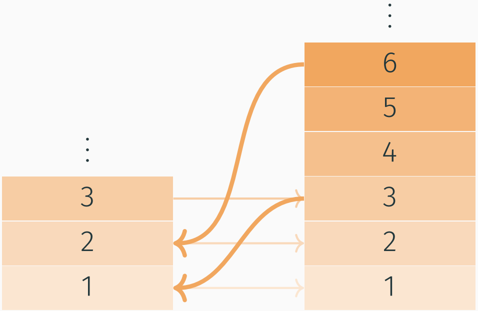

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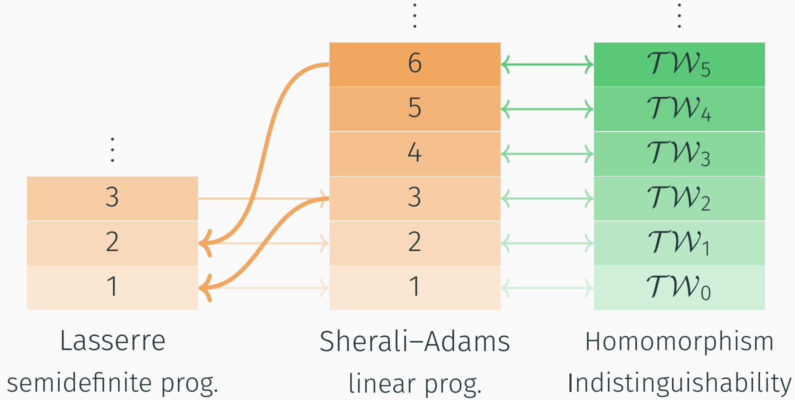

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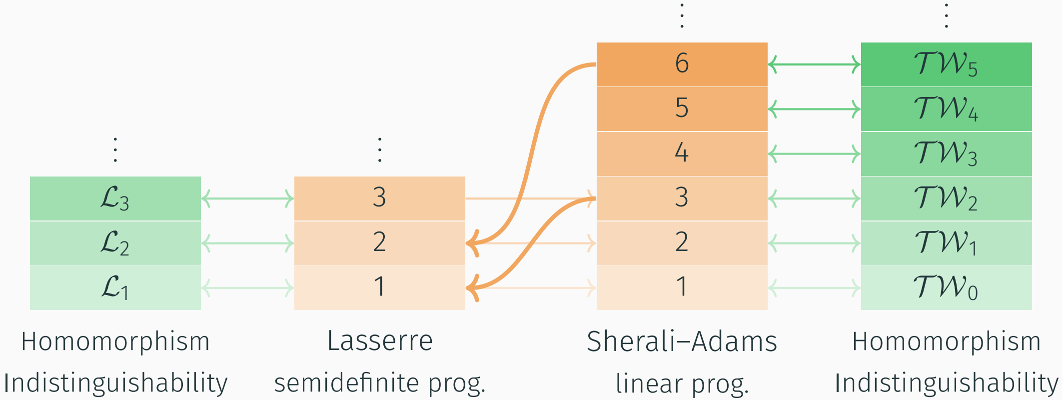

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Equations

homomorphism tensors,
algebraic operations,
simultaneous similarity

Graphs

(bi)labelled graphs,
combinatorial operations,
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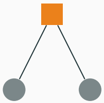
Labelled Graphs and Homomorphism Vectors



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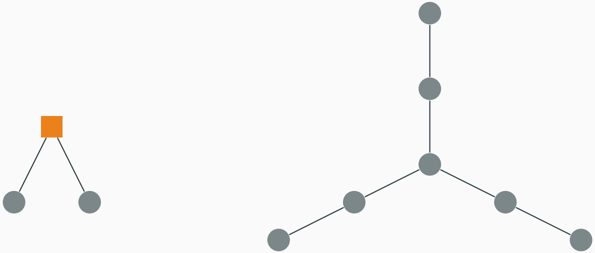

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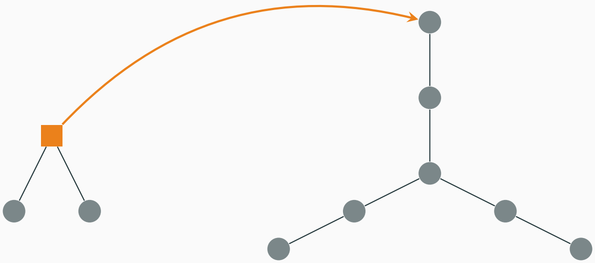

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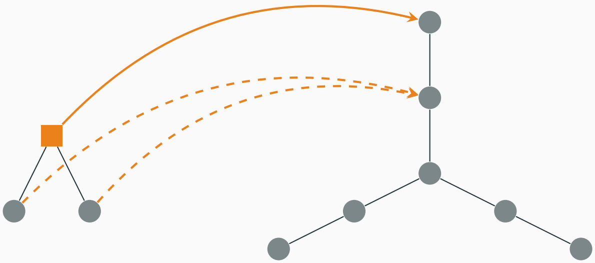

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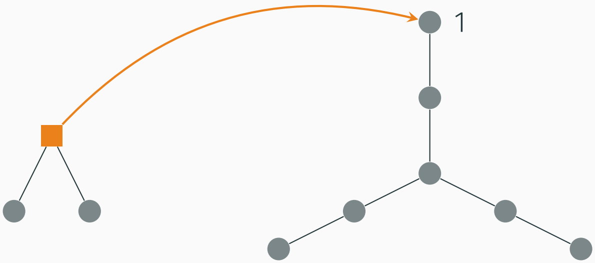

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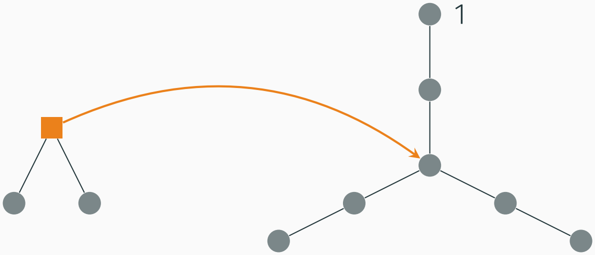

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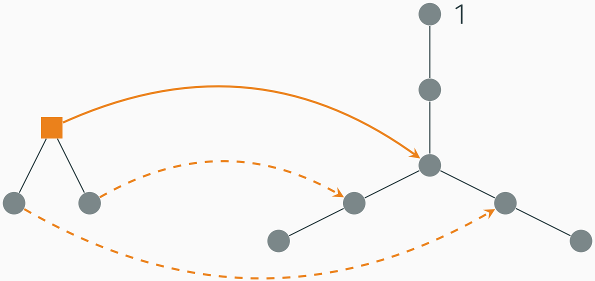

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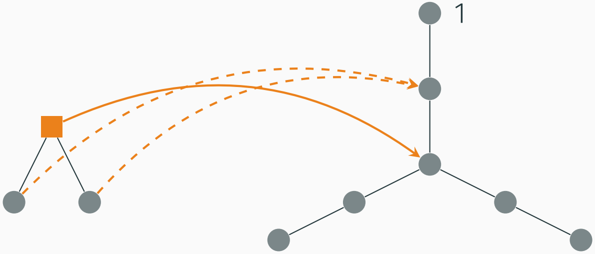

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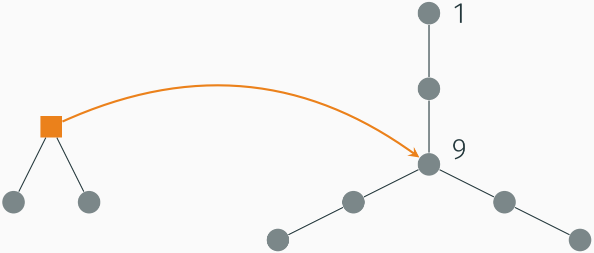
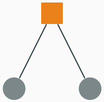
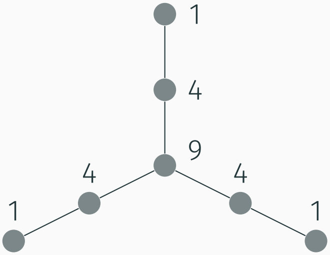


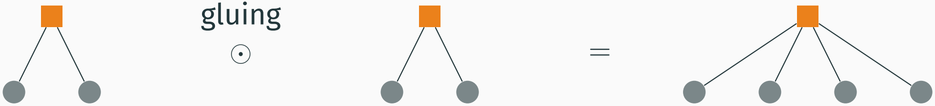
Labelled Graphs and Homomorphism Vectors

$$\mathcal{F} \longrightarrow \mathbb{C}^{V(G)}$$



$\mapsto$




gluing
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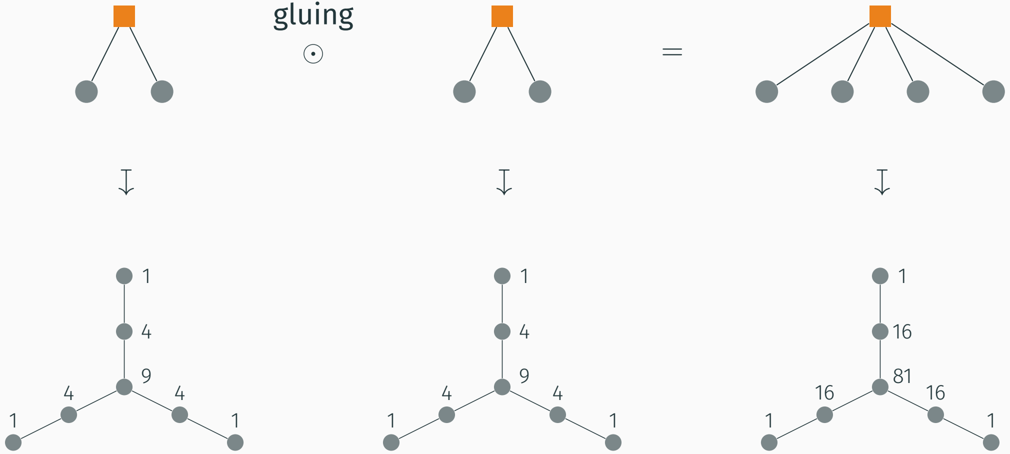
Combinatorial and Algebraic Operations



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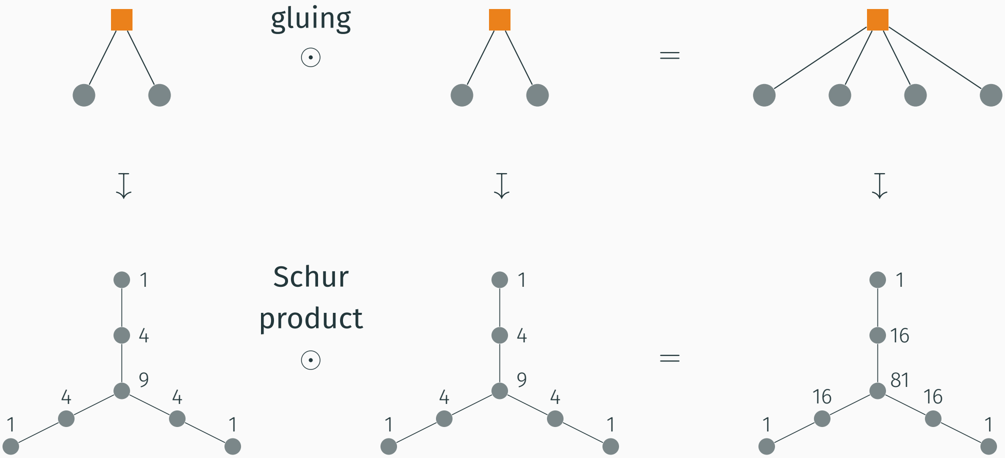

Simultaneous Similarity ⟷ Homomorphism Indistinguishability

Simultaneous Similarity $\longleftrightarrow$ Homomorphism Indistinguishability

Let $A_1, \dots, A_n$ and $B_1, \dots, B_n$ be complex square matrices.

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Theorem (Grohe, Rattan, S. (ICALP 2022))

X pseudo-stochastic s.t. $XA_i = B_i X$ ⟷ For every word $w \in [n]^*$, $\operatorname{soe} w_A = \operatorname{soe} w_B$.

Equations

homomorphism tensors,
algebraic operations,
simultaneous similarity

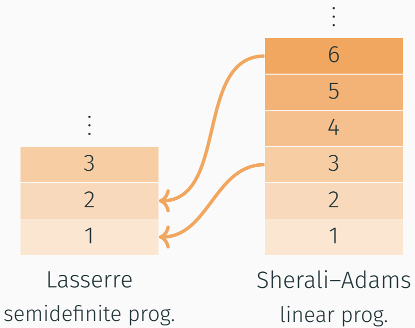
↔

Graphs

(bi)labelled graphs,
combinatorial operations,
homomorphism indist.

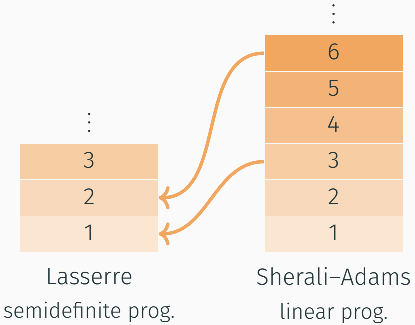
Upper Bound

- $\mathcal{L}_t \subseteq \mathcal{TW}_{3t-1}$,



Upper Bound

- $\mathcal{L}_t \subseteq \mathcal{TW}_{3t-1}$,
- $\mathcal{L}_t$ contains the clique K_{3t}.



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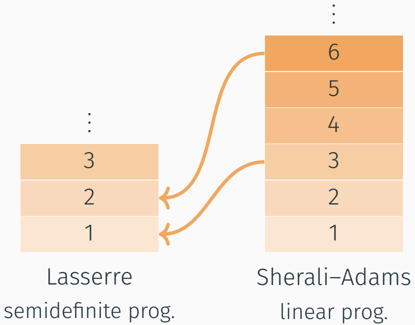

Lower Bound

$\mathcal{L}_t$ is a class of graphs of treewidth $\leq 3t - 1$ containing K_{3t}.

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Theorem (Neuen (2023))

$\mathcal{TW}_k$ is *homomorphism distinguishing closed.*

Corollary (Roberson and S. (ICALP 2023))

For every $t \geq 1$, there are graphs G and H such that $G \simeq^{SA}_{3t-1} H$ and $G \not\simeq^{L}_{t} H$.

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⋮

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2

1

Lasserre
semidefinite prog.

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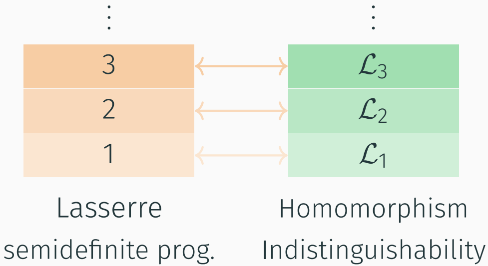

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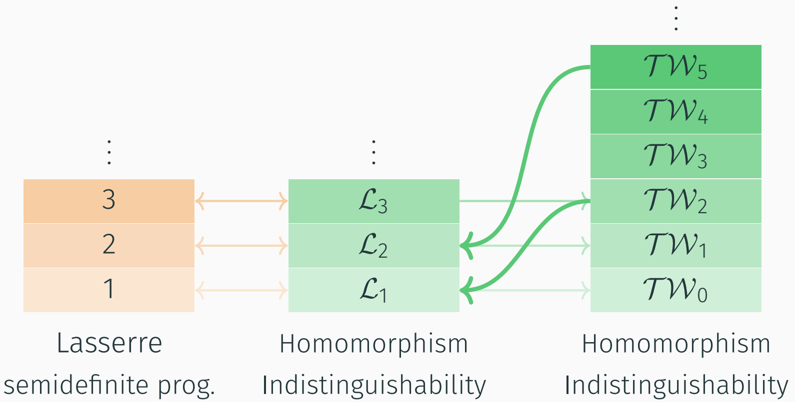

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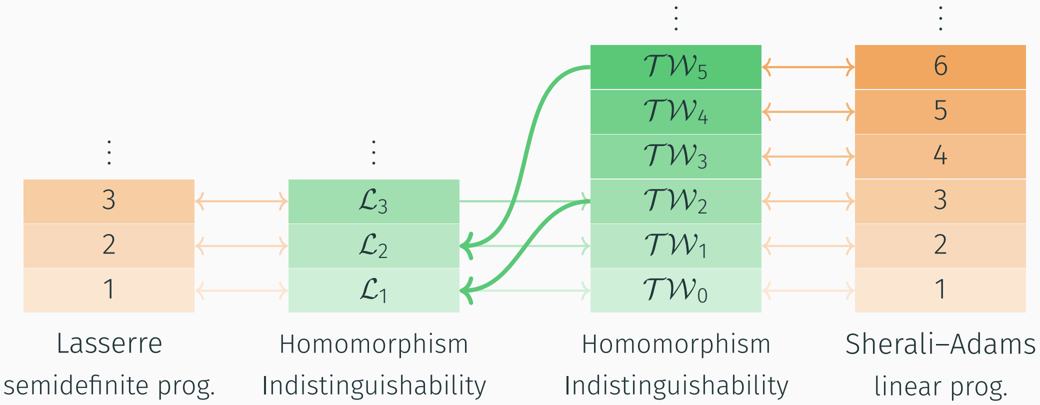
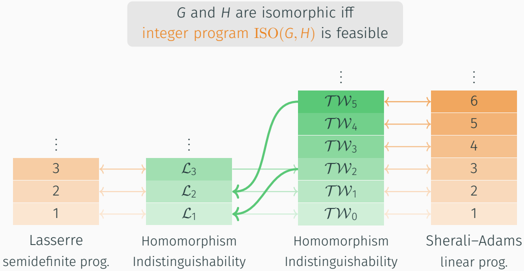


Conclusion

- Homomorphism indistinguishability characterisations of ISO relaxations



Conclusion

- Homomorphism indistinguishability characterisations of ISO relaxations
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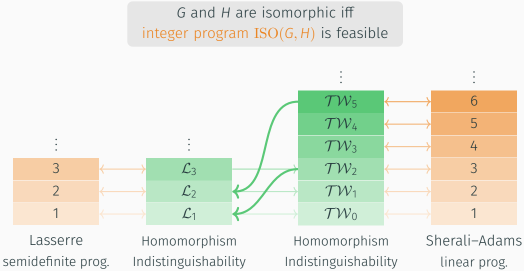
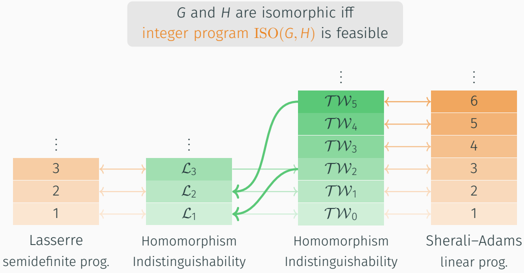


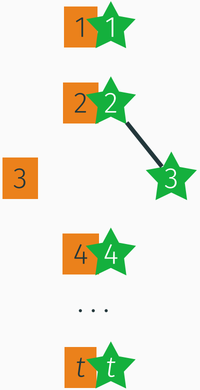
Conclusion

- Homomorphism indistinguishability characterisations of ISO relaxations
- Homomorphism tensors of (bi)labelled graphs
- Determined number of Sherali–Adams levels necessary to guarantee feasibility of Lasserre



The Graph Class $\mathcal{L}_t$

A (t, t)-bilabelled graph is *atomic* if all its vertices are labelled.

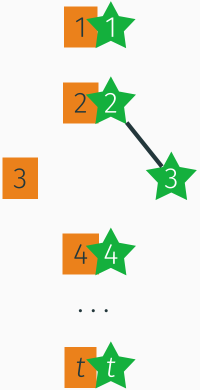


The Graph Class $\mathcal{L}_t$

A (t, t)-bilabelled graph is *atomic* if all its vertices are labelled.

The class $\mathcal{L}_t$ is generated by atomic graphs under

- series composition,
- parallel composition with atomic graphs,
- permutation of labels.



Extras: Lasserre

Let $t \geq 1$. The *level-t Lasserre relaxation for graph isomorphism* has variables y_I ranging over $\mathbb{R}$ for $I \in \binom{V(G)\times V(H)}{\leq 2t}$. The constraints are

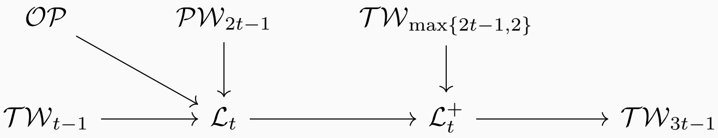
$$
\begin{aligned}
M_t(y) := (y_{I\cup J})_{I,J\in\binom{V(G)\times V(H)}{\leq t}} &\succeq 0, \\
\sum_{h\in V(H)} y_{I\cup\{gh\}} &= y_I \text{ for all } I \text{ s.t. } |I| \leq 2t-2 \text{ and all } g \in V(G), \\
\sum_{g\in V(G)} y_{I\cup\{gh\}} &= y_I \text{ for all } I \text{ s.t. } |I| \leq 2t-2 \text{ and all } h \in V(H), \\
y_I &= 0 \text{ if } I \text{ s.t. } |I| \leq 2t \text{ is not partial isomorphism} \\
y_\emptyset &= 1.
\end{aligned}
$$

Extras: Sherali–Adams

Let $t \geq 1$. The *level-t Sherali–Adams relaxation for graph isomorphism* has variables y_I ranging over $\mathbb{R}$ for $I \in \binom{V(G)\times V(H)}{\leq t}$. The constraints are

$$\sum_{h \in V(H)} y_{I \cup \{gh\}} = y_I \text{ for all } I \text{ s.t. } |I| \leq t-1 \text{ and all } g \in V(G),$$
$$\sum_{g \in V(G)} y_{I \cup \{gh\}} = y_I \text{ for all } I \text{ s.t. } |I| \leq t-1 \text{ and all } h \in V(H),$$
$$y_I = 0 \text{ if } I \text{ s.t. } |I| \leq t \text{ is not partial isomorphism}$$
$$y_\emptyset = 1.$$

Extra: Graph Classes



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Bibliography iv

Picture: “Bicycle race scene. A peloton of six cyclists crosses the finish line in front of a crowded grandstand, observed by a referee.” (1895) by Calvert Lithographic Co., Detroit, Michigan, Public Domain, via Wikimedia Commons. https://commons.wikimedia.org/wiki/File:Bicycle_race_scene,_1895.jpg