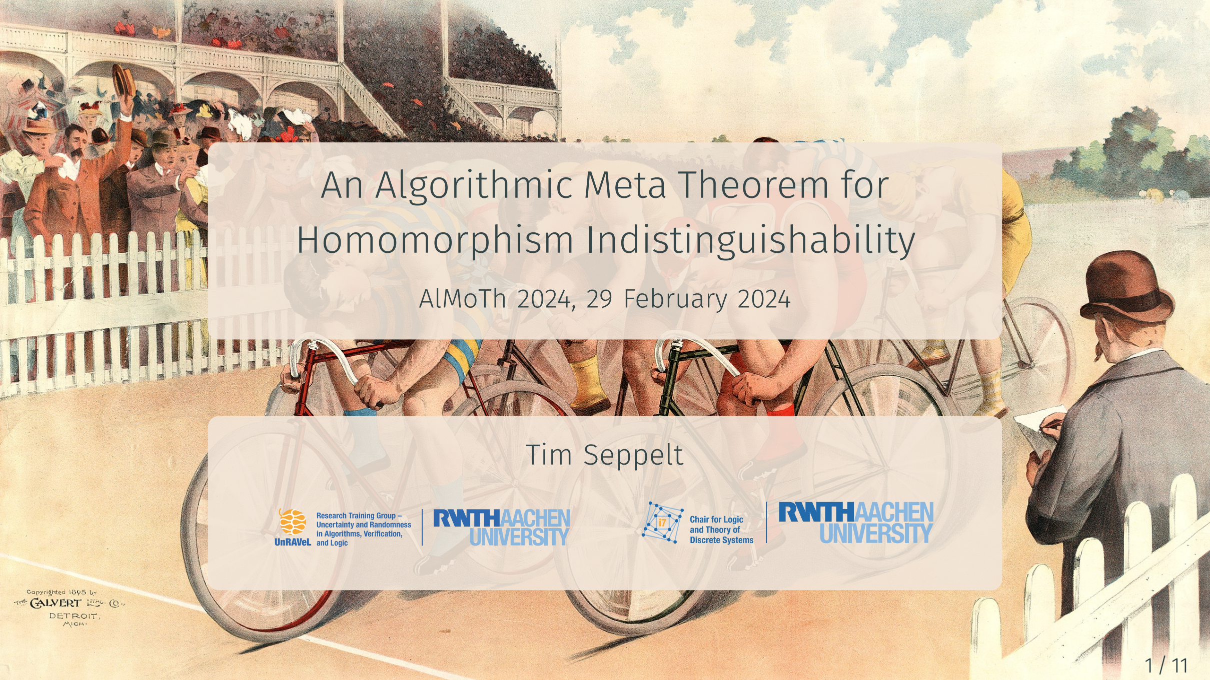


An Algorithmic Meta Theorem for Homomorphism Indistinguishability

AlMoTh 2024, 29 February 2024

Tim Seppelt

Research Training Group – Uncertainty and Randomness in Algorithms, Verification, and Logic
UnRAVeL
RWTH AACHEN UNIVERSITY

Chair for Logic and Theory of Discrete Systems
RWTH AACHEN UNIVERSITY

Complexity of Homomorphism Indistinguishability

HomInd($\mathcal{F}$)

Input Graphs G and H.

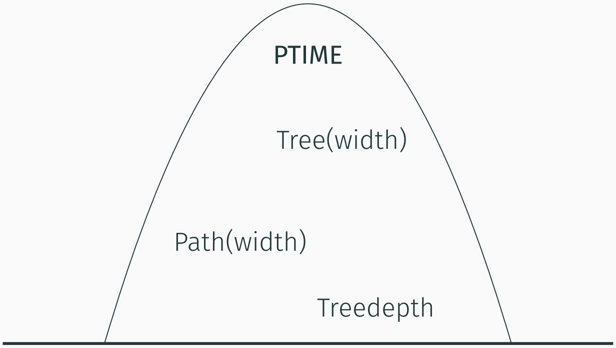
Decide $G \equiv_{\mathcal{F}} H$.

Complexity of Homomorphism Indistinguishability

HomInd($\mathcal{F}$)

Input Graphs G and H.

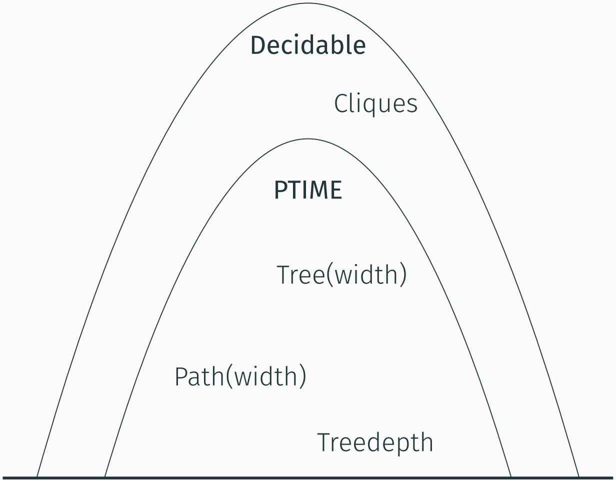
Decide $G \equiv_{\mathcal{F}} H$.



HomInd($\mathcal{F}$)

Input Graphs G and H.

Decide $G \equiv_{\mathcal{F}} H$.



HomInd($\mathcal{F}$)

Input Graphs G and H.

Decide $G \equiv_{\mathcal{F}} H$.

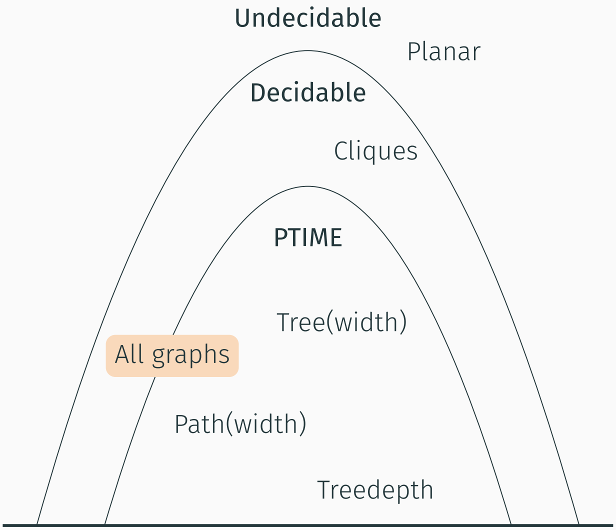
Complexity of Homomorphism Indistinguishability

Mančinska and Roberson (2020); Böker et al. (2019)

HomInd($\mathcal{F}$)

Input Graphs G and H.

Decide $G \equiv_{\mathcal{F}} H$.



An Algorithmic Meta Theorem

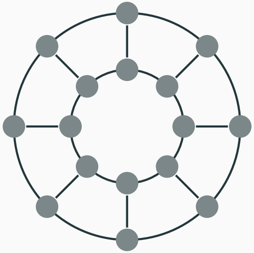
Theorem (S. (2024))

For every recognisable graph class $\mathcal{F}$ of bounded treewidth, $\textsc{HomInd}(\mathcal{F})$ *is in* coRP.

An Algorithmic Meta Theorem

Theorem (S. (2024))

For every recognisable graph class $\mathcal{F}$ of bounded treewidth, HomInd($\mathcal{F}$) *is in* coRP.

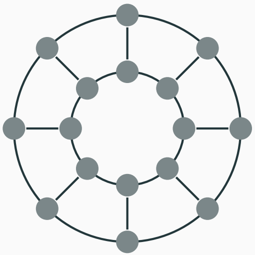


Minor-closed and bounded treewidth

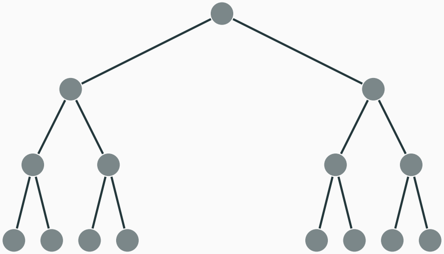
An Algorithmic Meta Theorem

Theorem (S. (2024))

For every recognisable graph class $\mathcal{F}$ of bounded treewidth, HOMIND($\mathcal{F}$) *is in* coRP.

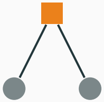


Minor-closed and bounded treewidth

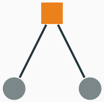


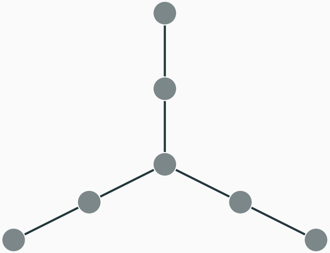
CMSO_2-definable and bounded treewidth

Courcelle (1990); Bojańczyk and Pilipczuk (2016)

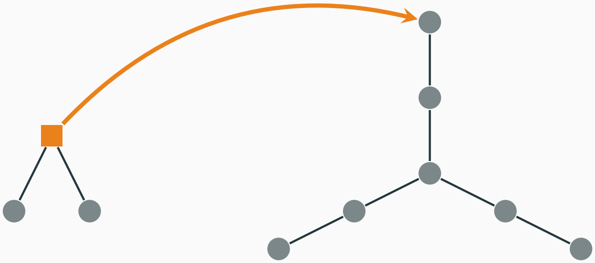


Labelled Graphs and Homomorphism Vectors

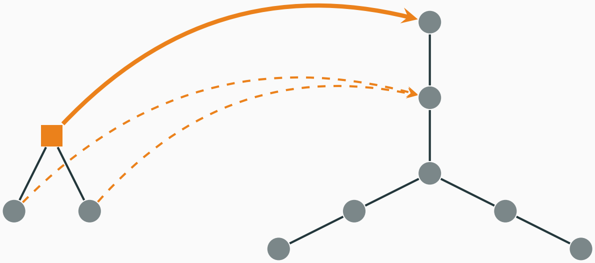




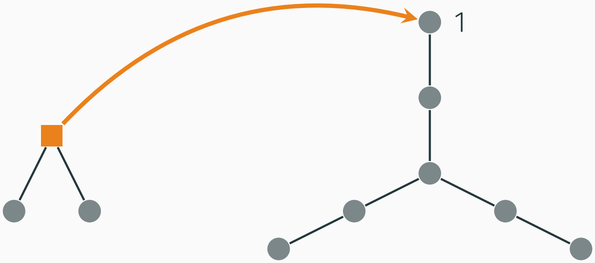
Labelled Graphs and Homomorphism Vectors



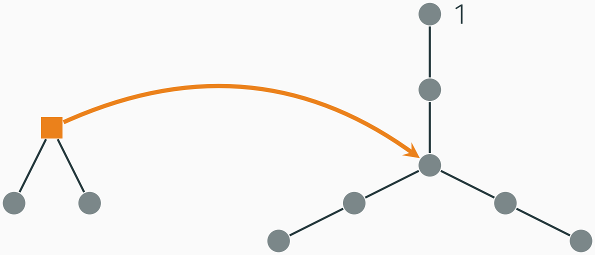
Labelled Graphs and Homomorphism Vectors



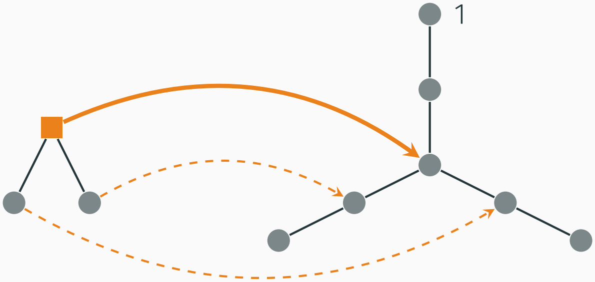
Labelled Graphs and Homomorphism Vectors



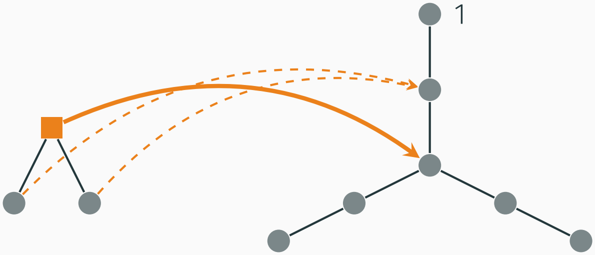
Labelled Graphs and Homomorphism Vectors



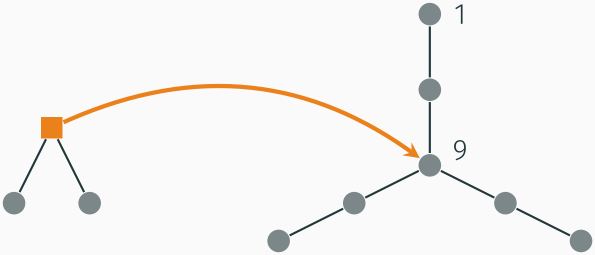
Labelled Graphs and Homomorphism Vectors



Labelled Graphs and Homomorphism Vectors

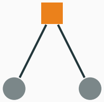


Labelled Graphs and Homomorphism Vectors

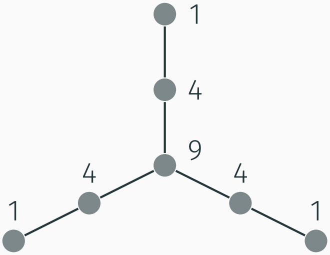


Labelled Graphs and Homomorphism Vectors

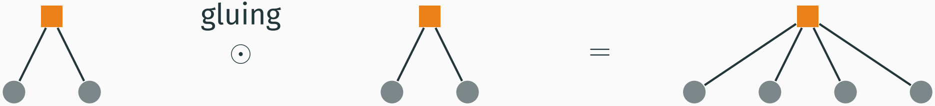
$$\mathcal{F} \longrightarrow \mathbb{R}^{V(G)}$$



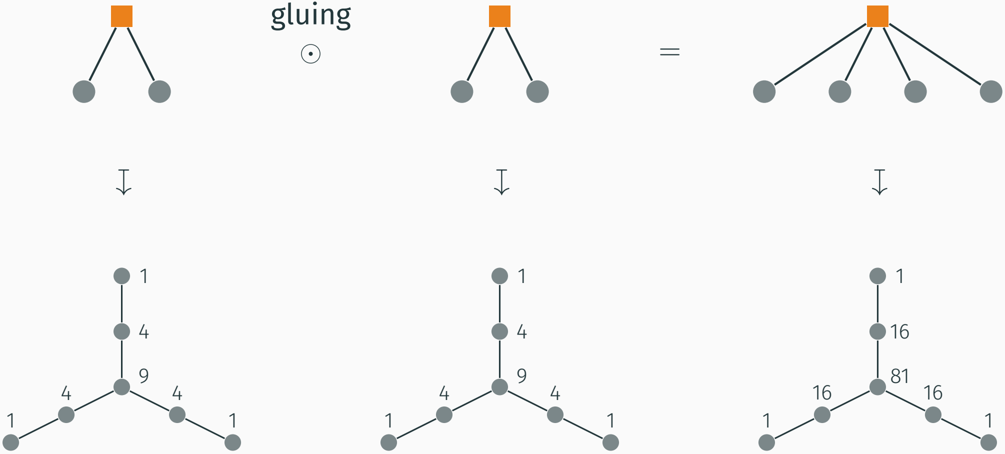
$\mapsto$



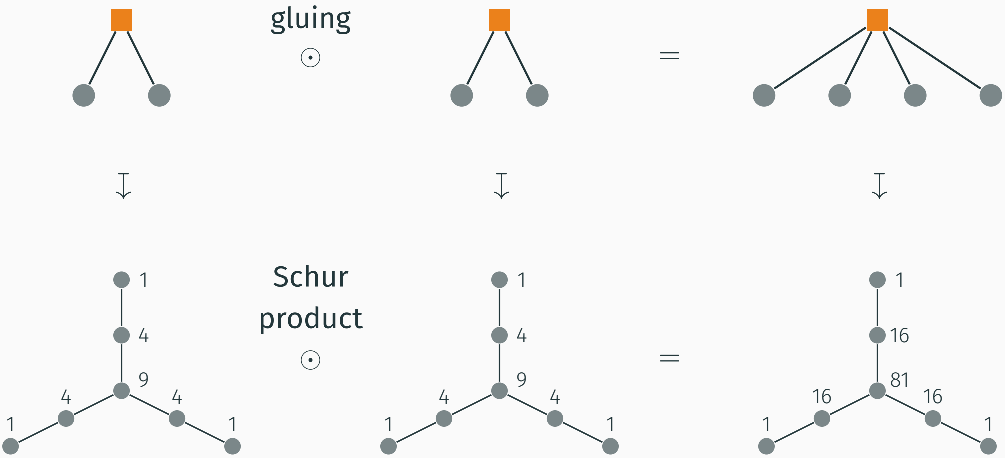
Combinatorial and Algebraic Operations: Gluing and Schur Product



Combinatorial and Algebraic Operations: Gluing and Schur Product



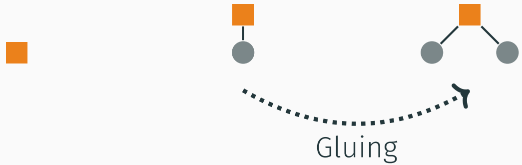
Combinatorial and Algebraic Operations: Gluing and Schur Product



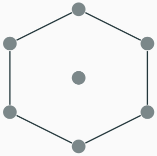
Homomorphism Indistinguishability over Trees

Homomorphism Indistinguishability over Trees

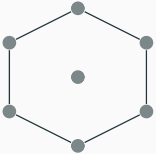
Attaching new edge


Gluing

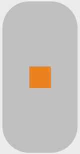
Homomorphism Indistinguishability over Trees

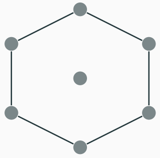


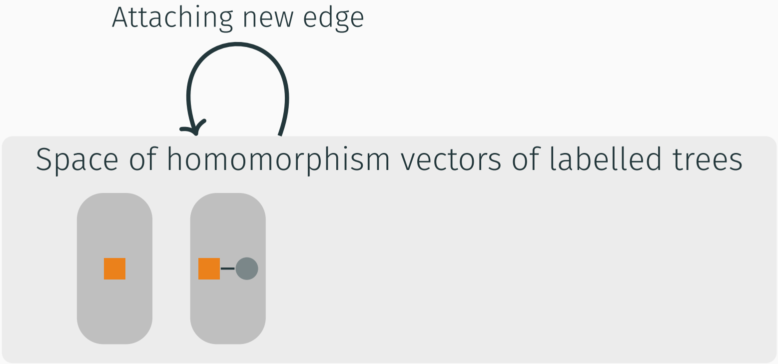
Space of homomorphism vectors of labelled trees



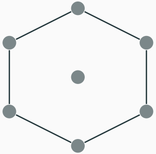
Space of homomorphism vectors of labelled trees

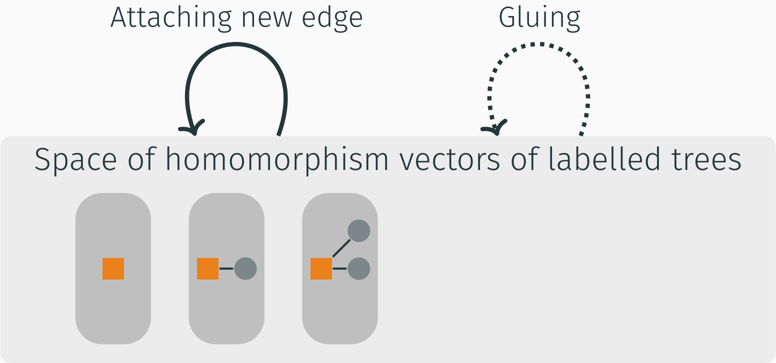




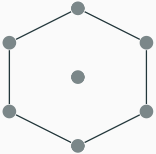


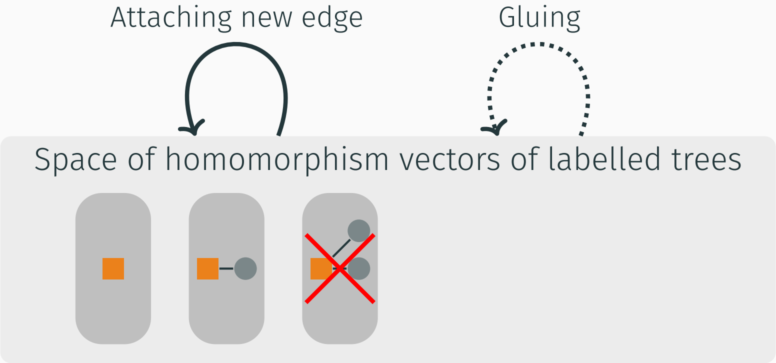
Attaching new edge
Space of homomorphism vectors of labelled trees

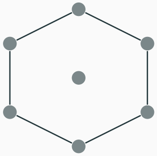


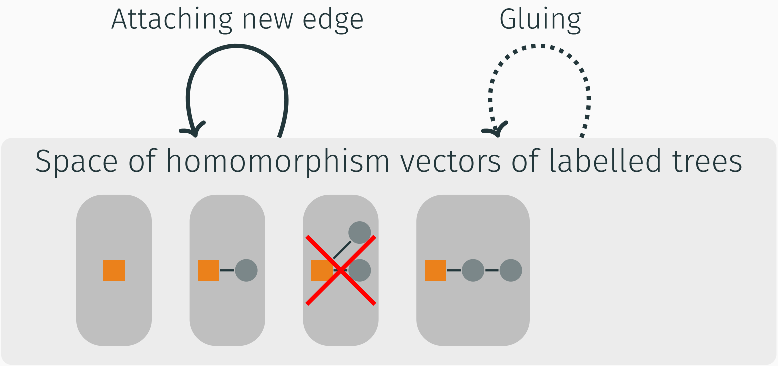


Attaching new edge
Gluing
Space of homomorphism vectors of labelled trees

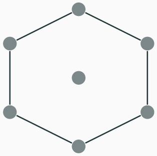


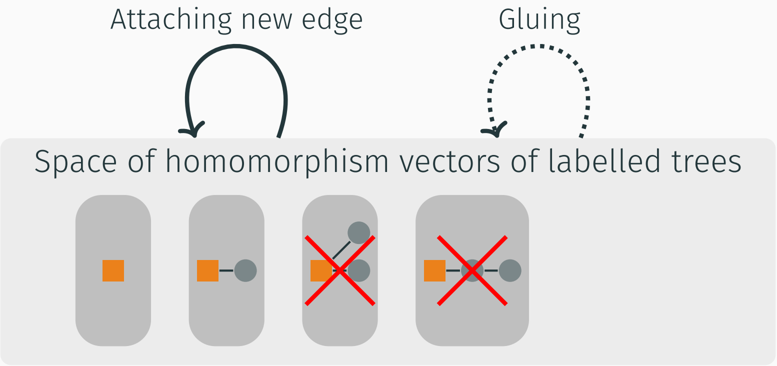

Attaching new edge
Gluing
Space of homomorphism vectors of labelled trees

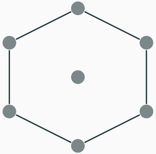


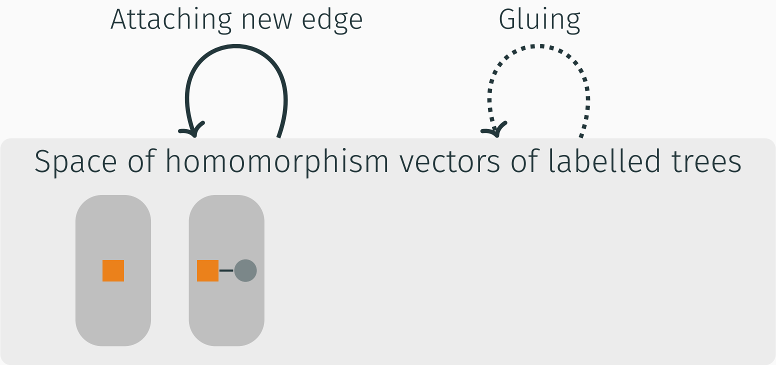


Attaching new edge
Gluing
Space of homomorphism vectors of labelled trees

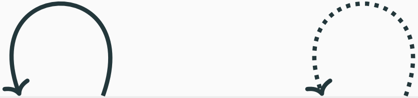



Attaching new edge
Gluing
Space of homomorphism vectors of labelled trees

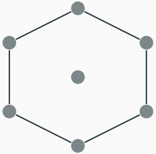


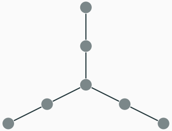

Attaching new edge
Gluing
Space of homomorphism vectors of labelled trees

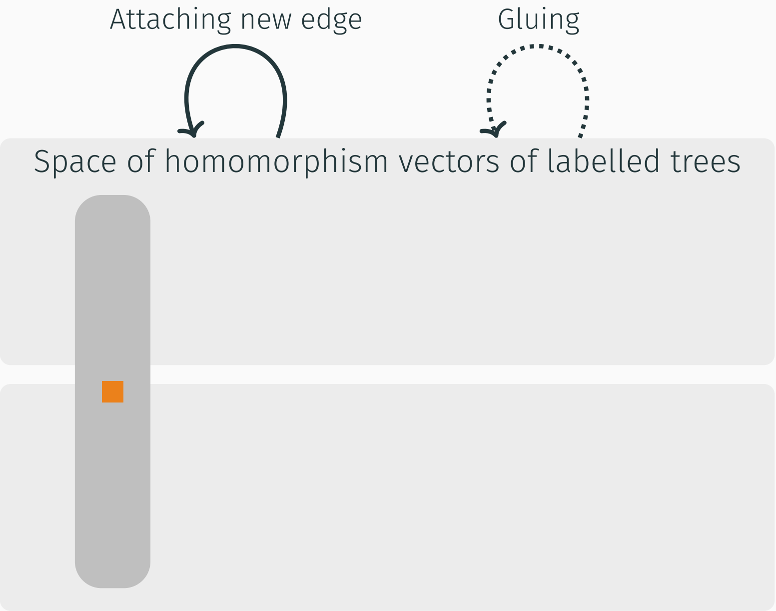
Attaching new edge Gluing

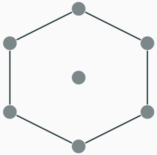


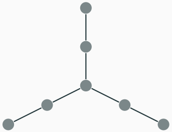
Space of homomorphism vectors of labelled trees

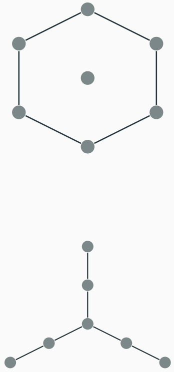


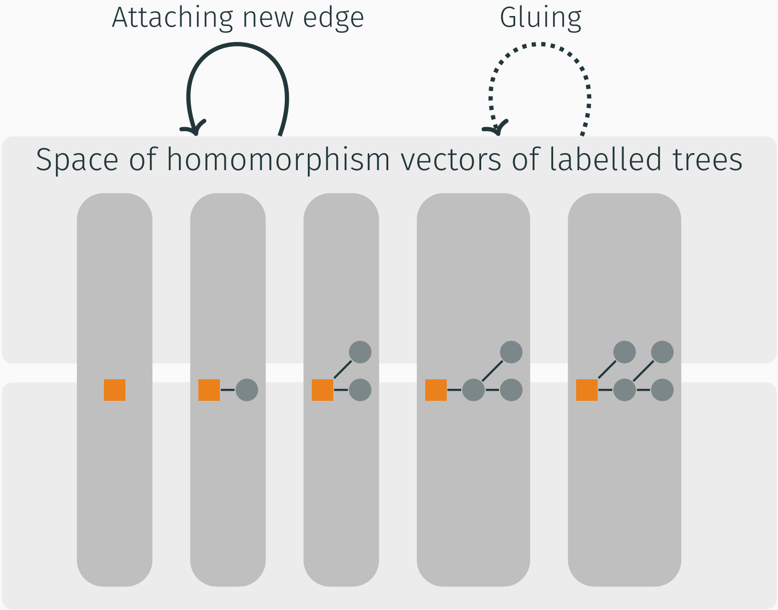



Attaching new edge
Gluing
Space of homomorphism vectors of labelled trees

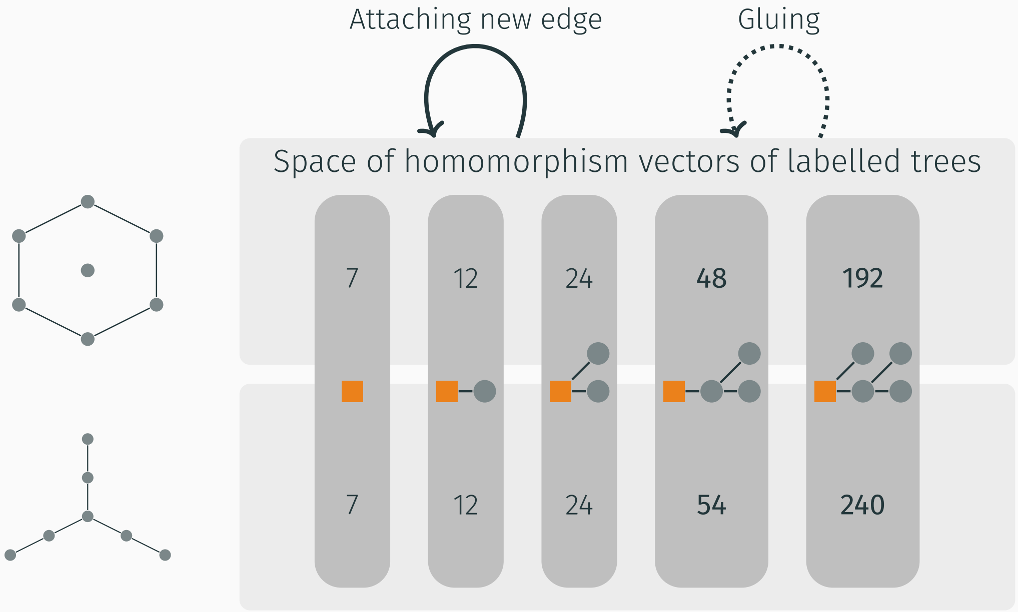








Attaching new edge
Gluing
Space of homomorphism vectors of labelled trees


Attaching new edge
Gluing
Space of homomorphism vectors of labelled trees
7
12
24
48
192
7
12
24
54
240

Recognisability

Courcelle (1990)

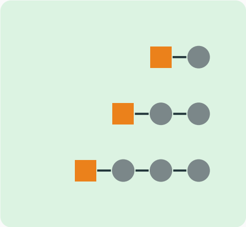
Let $\mathcal{F}$ be a graph class. Let $F_1 \sim_{\mathcal{F}} F_2$ iff for all labelled graphs K

$$\mathrm{unlabel}(K \odot F_1) \in \mathcal{F} \iff \mathrm{unlabel}(K \odot F_2) \in \mathcal{F}.$$

Recognisability

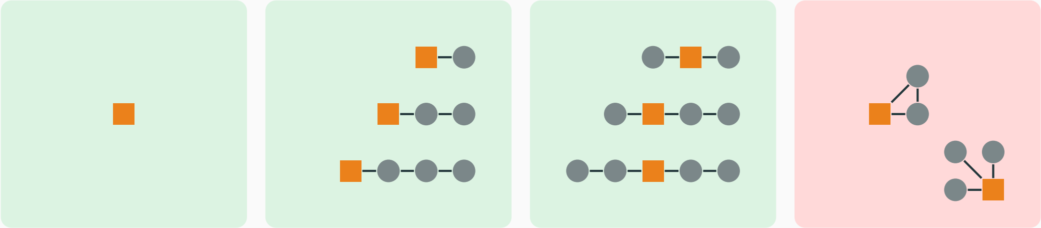
Let $\mathcal{F}$ be a graph class. Let $F_1 \sim_{\mathcal{F}} F_2$ iff for all labelled graphs K

$$\mathrm{unlabel}(K \odot F_1) \in \mathcal{F} \iff \mathrm{unlabel}(K \odot F_2) \in \mathcal{F}.$$



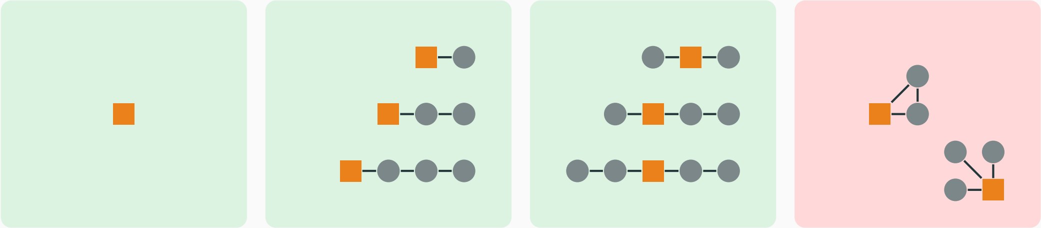
Let $\mathcal{F}$ be a graph class. Let $F_1 \sim_{\mathcal{F}} F_2$ iff for all labelled graphs K

$$\mathrm{unlabel}(K \odot F_1) \in \mathcal{F} \iff \mathrm{unlabel}(K \odot F_2) \in \mathcal{F}.$$



Let $\mathcal{F}$ be a graph class. Let $F_1 \sim_{\mathcal{F}} F_2$ iff for all labelled graphs K

$$\text{unlabel}(K \odot F_1) \in \mathcal{F} \iff \text{unlabel}(K \odot F_2) \in \mathcal{F}.$$

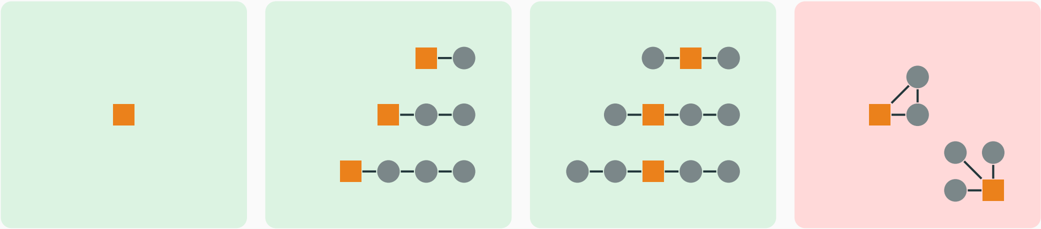


Definition

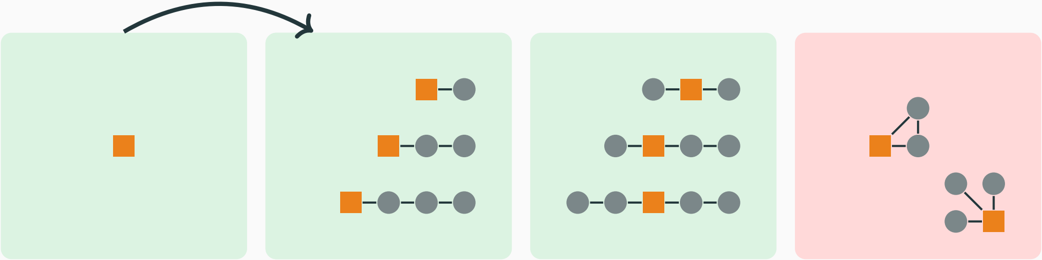
$\mathcal{F}$ is *recognisable* if $\sim_{\mathcal{F}}$ has finitely many equivalence classes.

Attaching a new edge

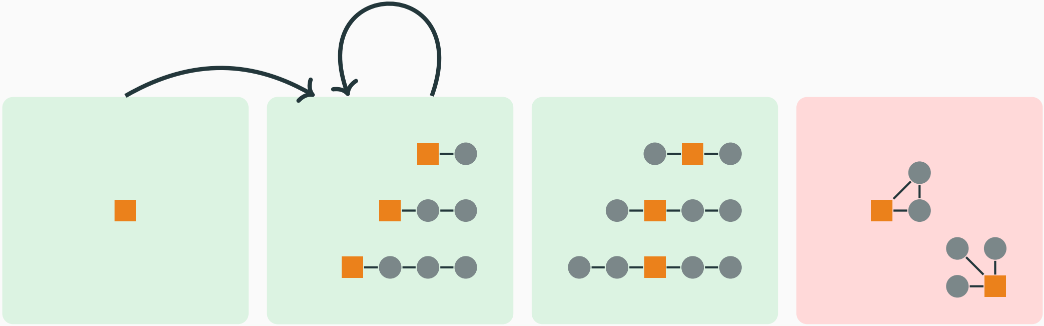
Attaching a new edge



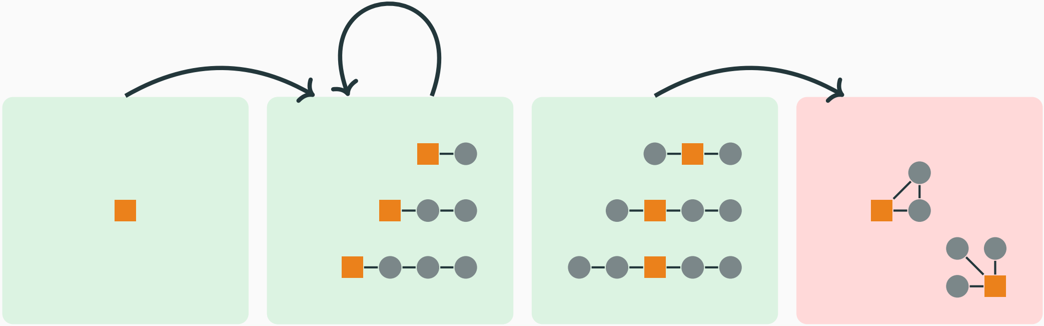
Attaching a new edge



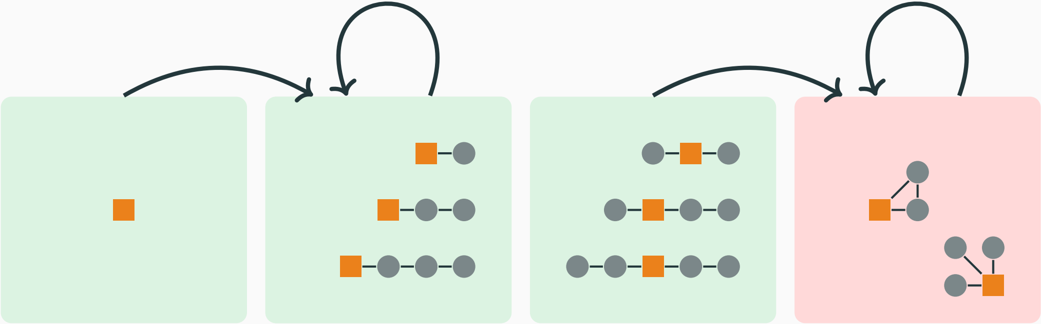
Attaching a new edge

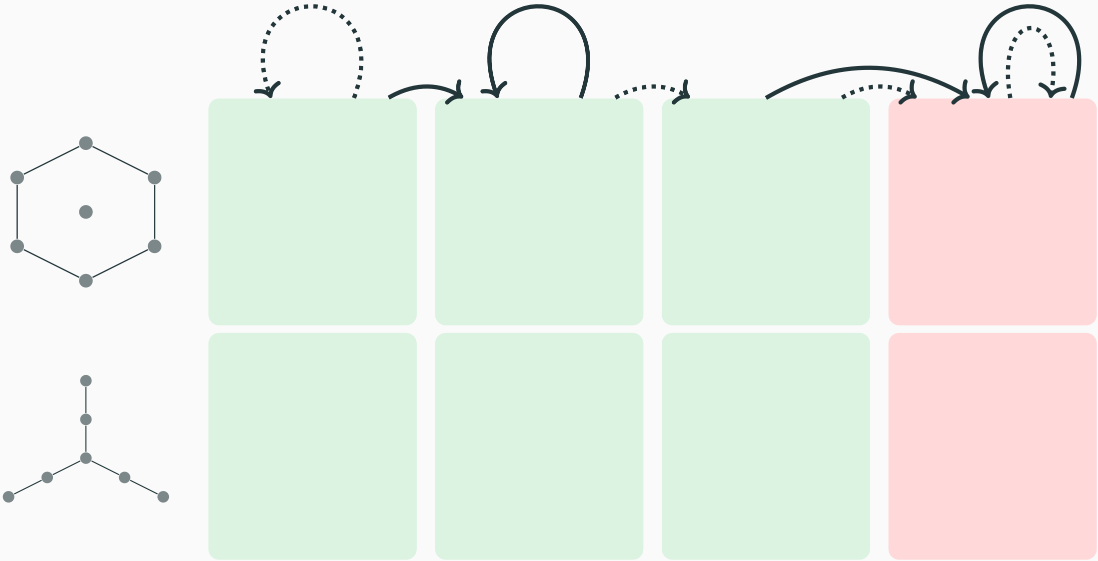


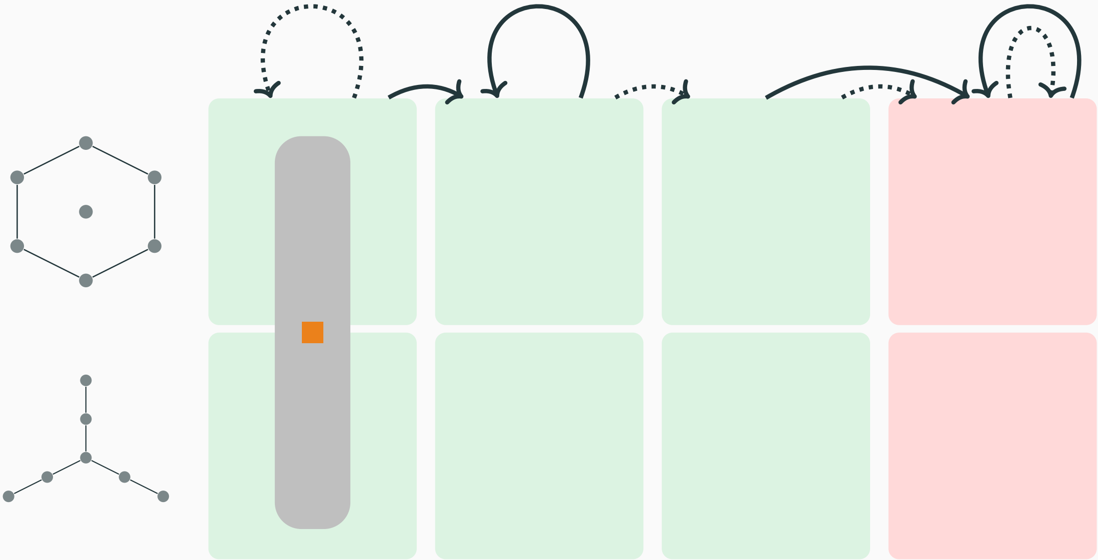
Attaching a new edge

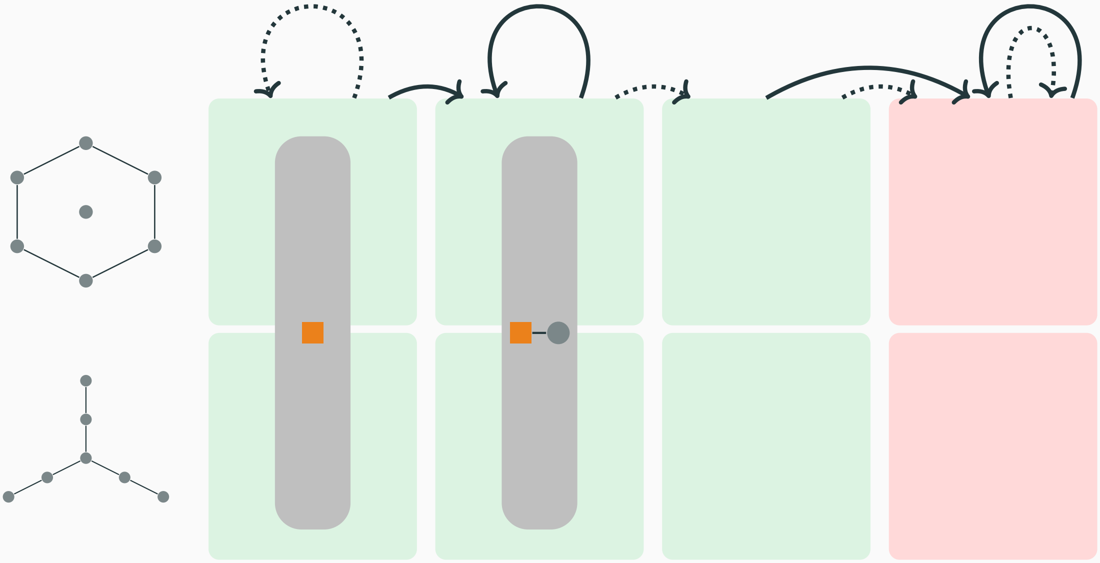


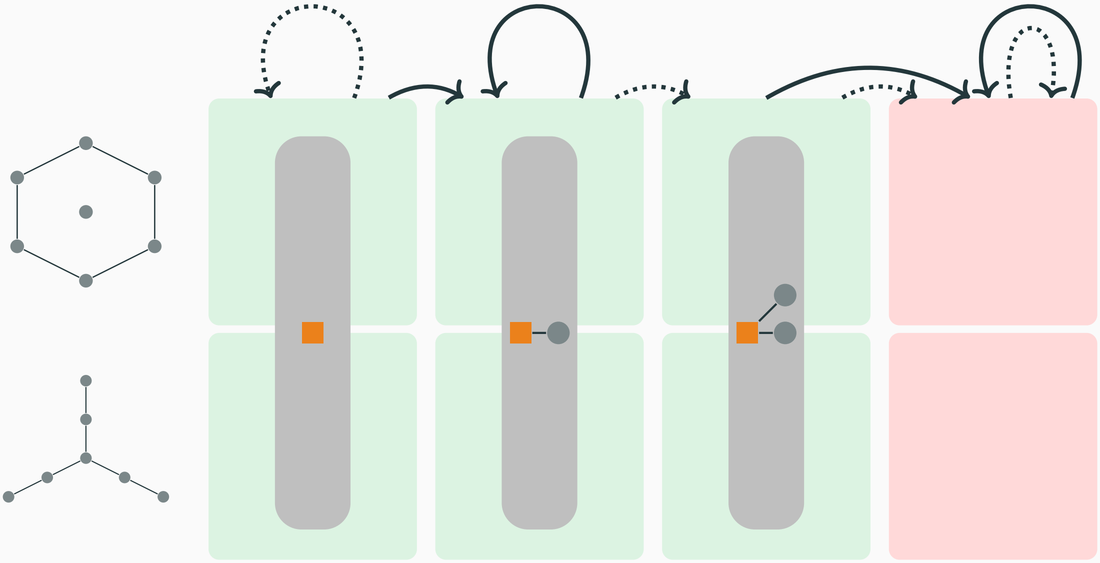
Attaching a new edge

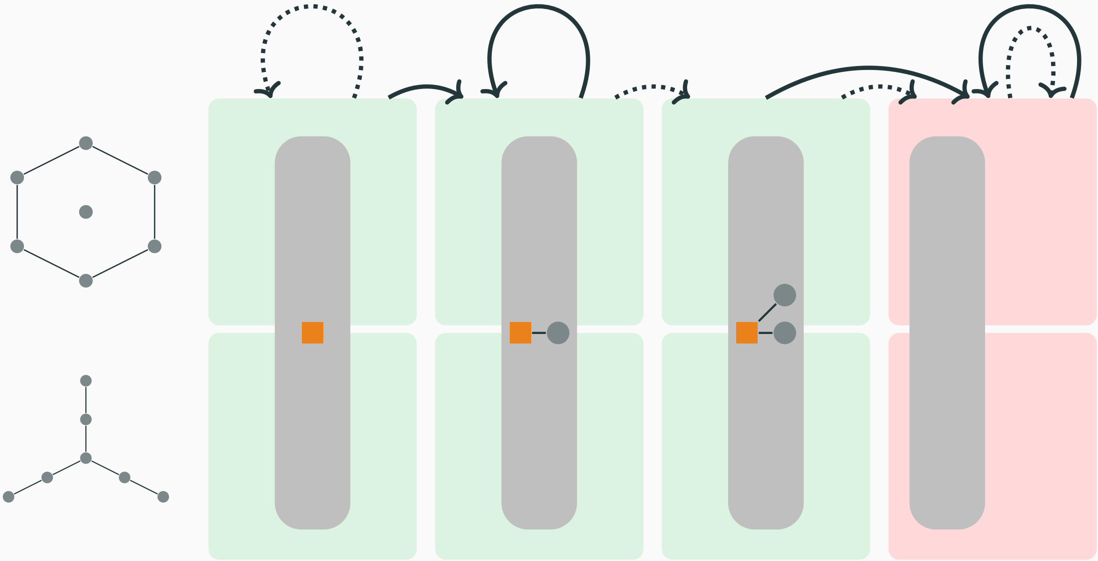


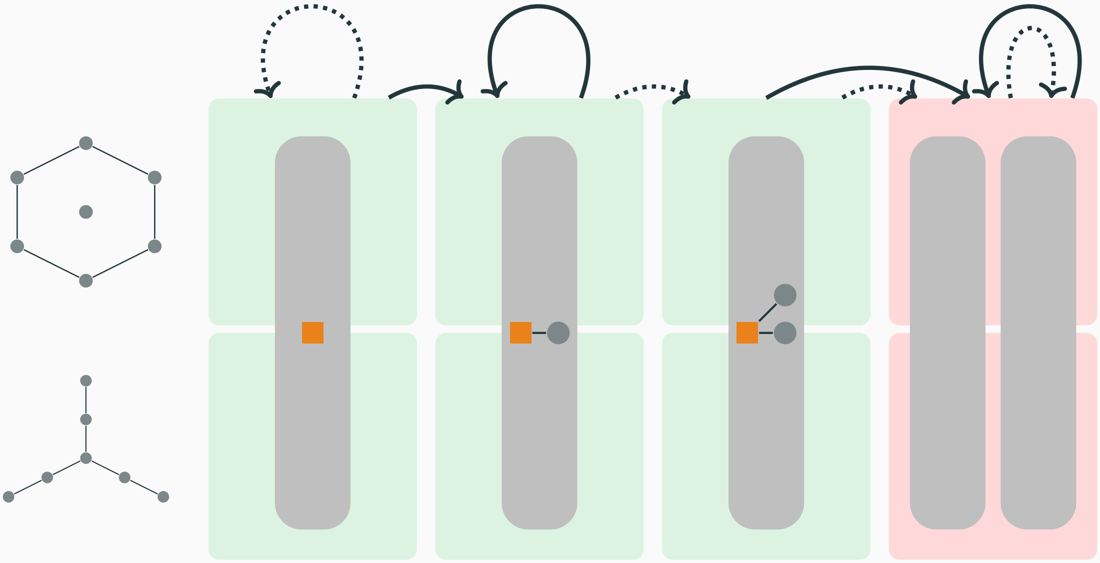


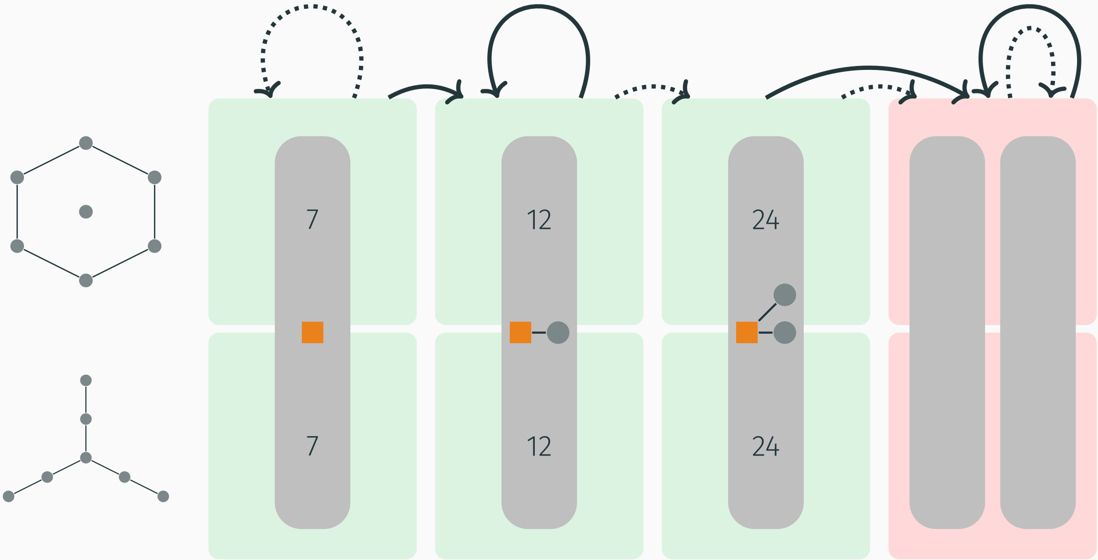










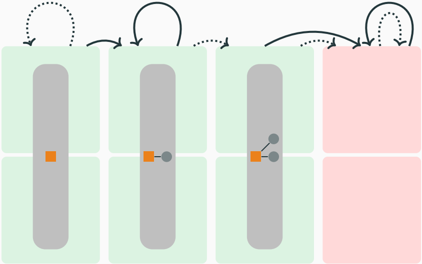

7
7
12
12
24
24

Conclusion

Theorem (S. (2024))

For every recognisable graph class $\mathcal{F}$ of bounded treewidth, HomInd($\mathcal{F}$) *is in* coRP.

- Randomised fixed-parameter algorithm for CMSO_2-sentence specifying $\mathcal{F}$.



A conjecture

Conjecture

For a minor-closed $\mathcal{F}$, either

$\mathcal{F}$ *contains all graphs* *and* $\textsc{HomInd}(\mathcal{F})$ *is Graph Isomorphism*,

A conjecture

Conjecture

For a minor-closed $\mathcal{F}$, either

$\mathcal{F}$ *contains all graphs*	*and*	$\textsc{HomInd}(\mathcal{F})$ *is Graph Isomorphism*,
$\mathcal{F}$ *has bounded treewidth*	*and*	$\textsc{HomInd}(\mathcal{F})$ *is in* coRP, *or*

A conjecture

Conjecture

For a minor-closed $\mathcal{F}$, either

$\mathcal{F}$ *contains all graphs*	*and*	$\textsc{HomInd}(\mathcal{F})$ *is Graph Isomorphism,*
$\mathcal{F}$ *has bounded treewidth*	*and*	$\textsc{HomInd}(\mathcal{F})$ *is in coRP, or*
$\mathcal{F}$ *has unbounded treewidth*	*and*	$\textsc{HomInd}(\mathcal{F})$ *is undecidable.*

Parametrised HomInd

HomInd

Input Graphs G and H, a CSMO_2-sentence φ, an integer $k \in \mathbb{N}$

Parameter $|\varphi| + k$

Decide Are G and H homomorphism indistinguishable over $\mathcal{F}_{\varphi,k} := \{F \mid \operatorname{tw} F \leq k,\ F \models \varphi\}$?

Parametrised HomInd

HomInd

Input Graphs G and H, a CSMO_2-sentence φ, an integer $k \in \mathbb{N}$

Parameter $|\varphi| + k$

Decide Are G and H homomorphism indistinguishable over $\mathcal{F}_{\varphi,k} \coloneqq \{F \mid \operatorname{tw} F \leq k,\ F \models \varphi\}$?

Theorem (S. (2024))

There is a randomised algorithm for HomInd *running in time $f(k, |\varphi|)n^{O(k)}$.*

Bojańczyk, Mikołaj and Michał Pilipczuk (July 2016). "Definability equals recognizability for graphs of bounded treewidth". In: *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science*. New York NY USA: ACM, pp. 407–416. ISBN: 978-1-4503-4391-6. DOI: 10.1145/2933575.2934508. URL: https://dl.acm.org/doi/10.1145/2933575.2934508.

Böker, Jan, Yijia Chen, Martin Grohe, and Gaurav Rattan (2019). "The Complexity of Homomorphism Indistinguishability". en. In: Artwork Size: 13 pages Medium: application/pdf Publisher: Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik GmbH, Wadern/Saarbruecken, Germany, 13 pages. DOI: 10.4230/LIPICS.MFCS.2019.54. URL: http://drops.dagstuhl.de/opus/volltexte/2019/10998/.

Courcelle, Bruno (Mar. 1990). "The monadic second-order logic of graphs. I. Recognizable sets of finite graphs". In: *Information and Computation* 85.1, pp. 12–75. ISSN: 0890-5401. DOI: 10.1016/0890-5401(90)90043-H. URL: https://www.sciencedirect.com/science/article/pii/089054019090043H.

Mančinska, Laura and David E. Roberson (2020). "Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs". In: *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*, pp. 661–672. DOI: 10.1109/FOCS46700.2020.00067.

Seppelt, Tim (2024). "An Algorithmic Meta Theorem for Homomorphism Indistinguishability". In: *CoRR* abs/2402.08989. DOI: 10.48550/ARXIV.2402.08989. arXiv: 2402.08989. URL: https://doi.org/10.48550/arXiv.2402.08989.

Title Picture: "Bicycle race scene. A peloton of six cyclists crosses the finish line in front of a crowded grandstand, observed by a referee." (1895) by Calvert Lithographic Co., Detroit, Michigan, Public Domain, via Wikimedia Commons. `https://commons.wikimedia.org/wiki/File:Bicycle_race_scene,_1895.jpg`