Homomorphism Indistinguishability

PhD Defense, 29 November 2024

Tim Seppelt

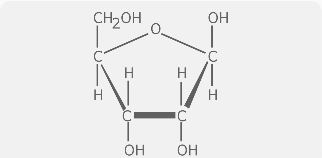
UnRAVeL Research Training Group – Uncertainty and Randomness in Algorithms, Verification, and Logic

i7 Chair for Logic and Theory of Discrete Systems

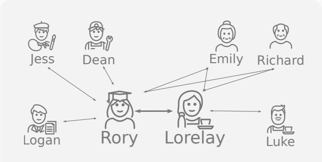
RWTH AACHEN UNIVERSITY

Funded by the European Union

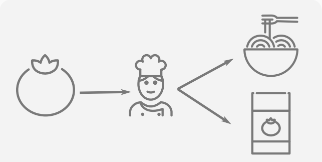
erc European Research Council



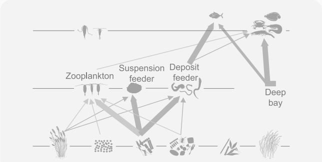
Chemical Compounds



Social Networks

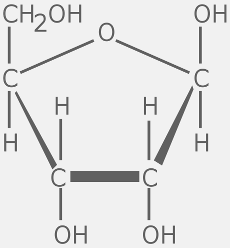


Program Executions



Trophic Networks

T.vanschaik, CC BY-SA 4.0, via Wikimedia Commons; Amy Sherman-Palladino (2000); OpenMoji (CC BY-SA 4.0); Kang et al. (2020)



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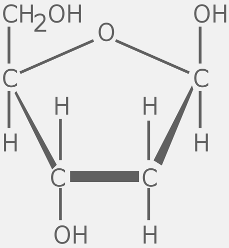

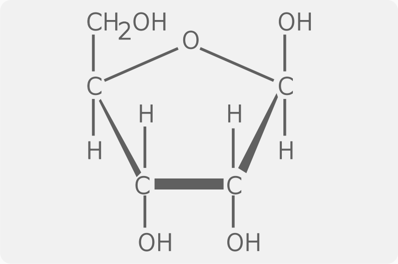

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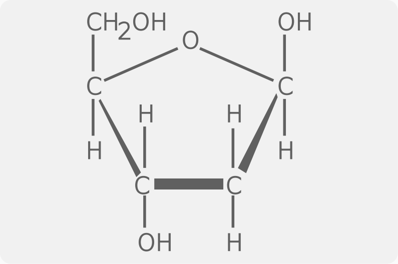
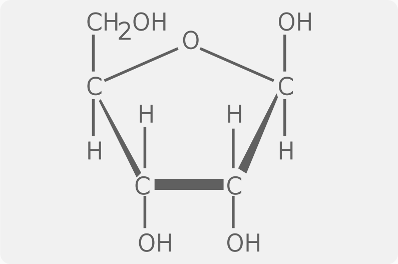
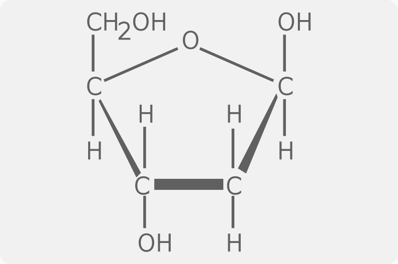


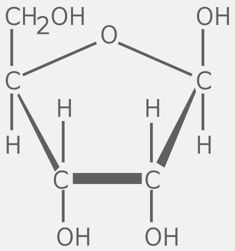


$\ncong$



However, graph isomorphism is

- theoretically elusive and
- practically often inconsequential.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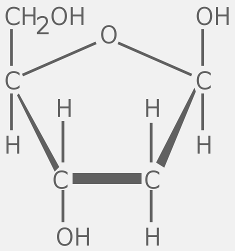
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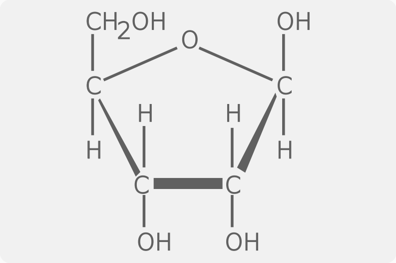
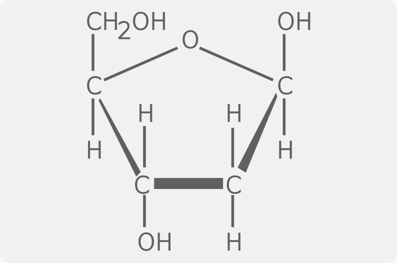
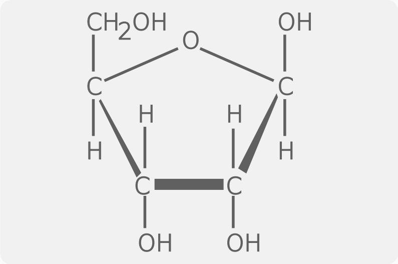
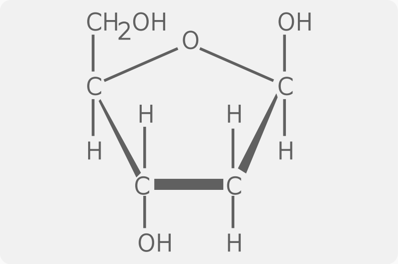


$\approx$



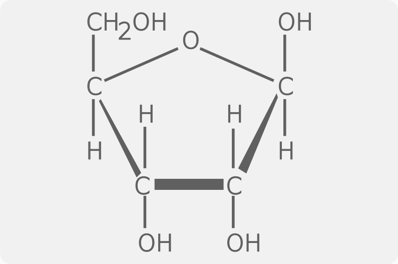


$\approx$

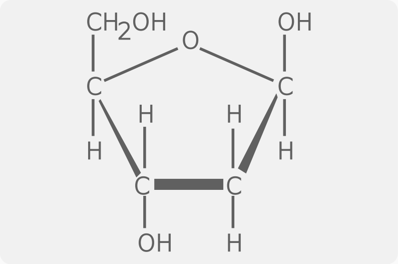


Characterisations

How to characterise $\approx$?



≈

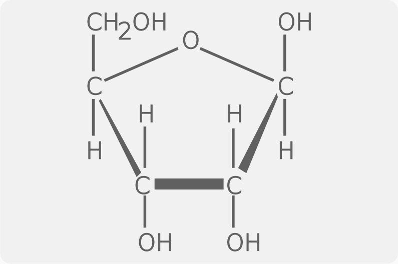


Characterisations

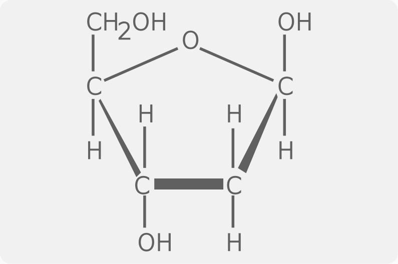
How to characterise ≈?

Distinguishing Power

What's the power of ≈?



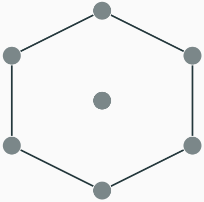
$\approx$

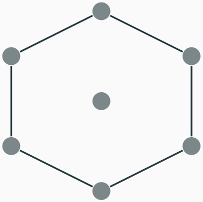


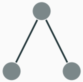
Characterisations
How to characterise $\approx$?

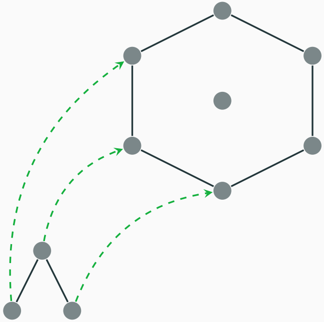
Distinguishing Power
What's the power of $\approx$?

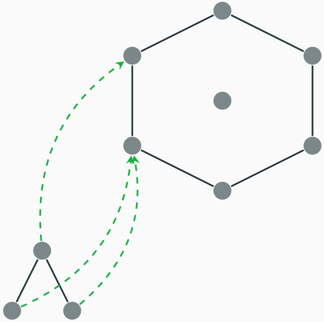
Complexity
How to test $\approx$?

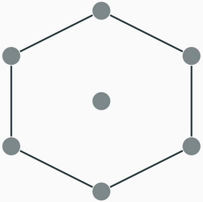


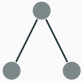












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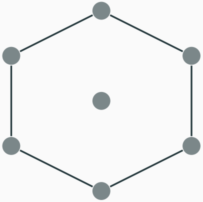

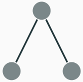

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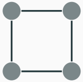

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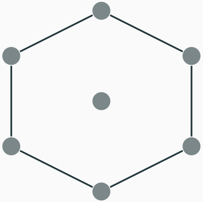
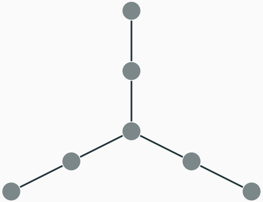
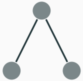
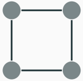
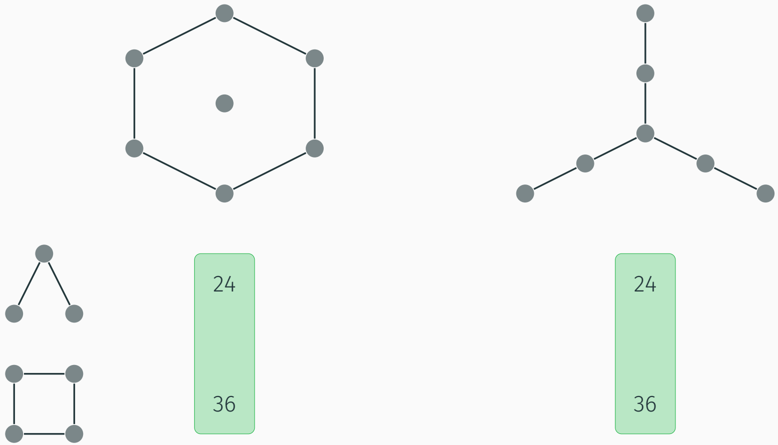
		
	24	24
	36	36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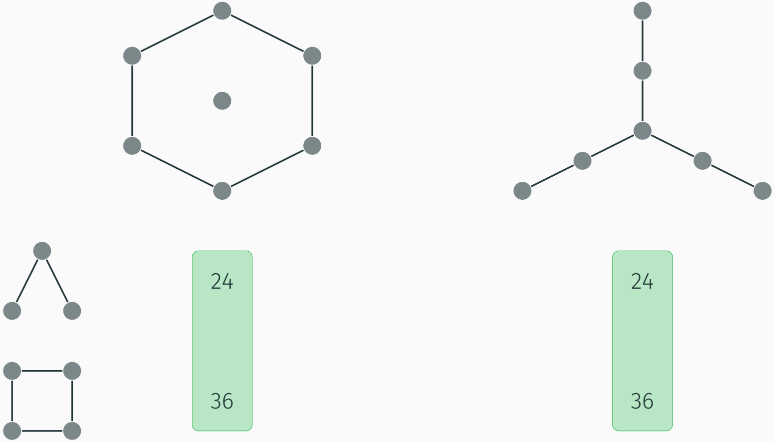
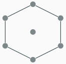
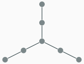


The graphs  and  are **homomorphism indistinguishable** over {, }.

graph class $\mathcal{F}$	**relation $\equiv_{\mathcal{F}}$**	
all graphs	isomorphism	Lovász (1967)

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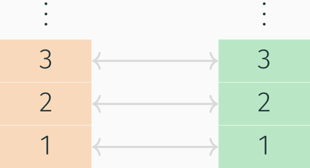
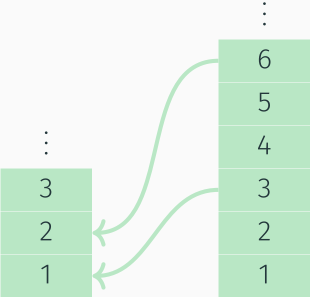
	machine learning	Xu et al. (2018); Morris et al. (2019)
...	...	



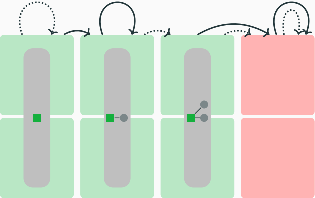
Characterisations

How to characterise $\equiv_{\mathcal{F}}$?



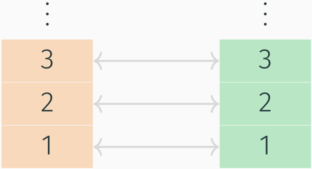
Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?



Complexity

How to test $\equiv_{\mathcal{F}}$?



Characterisations

How to characterise $\equiv_{\mathcal{F}}$?

Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?

Complexity

How to decide $\equiv_{\mathcal{F}}$?

Graphs G and H are isomorphic if and only if there is a permutation matrix X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$

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↓

positive semi-definite X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$.

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⋮
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Lasserre semidefinite prog.

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⋮

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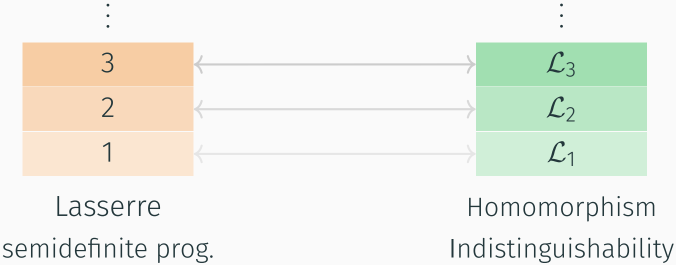

Roberson and S. (2023)

Equations
homomorphism vectors
algebraic operations

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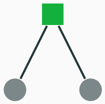
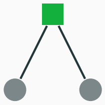
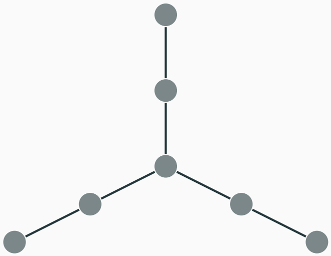
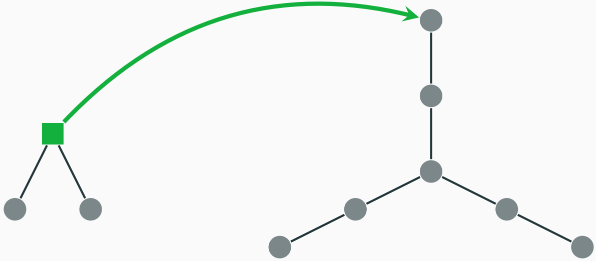
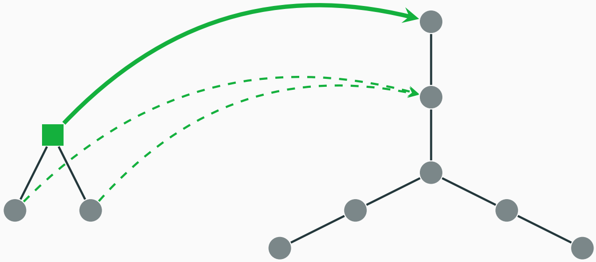
Graph Class
labelled graphs
combinatorial operations

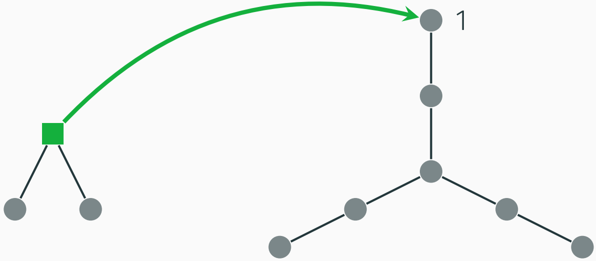








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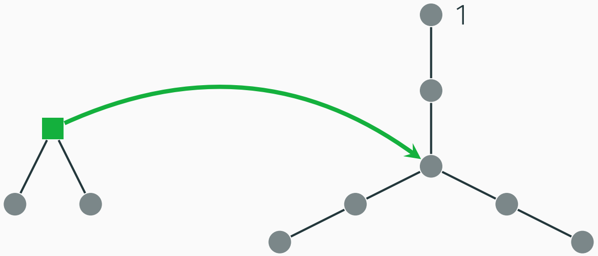
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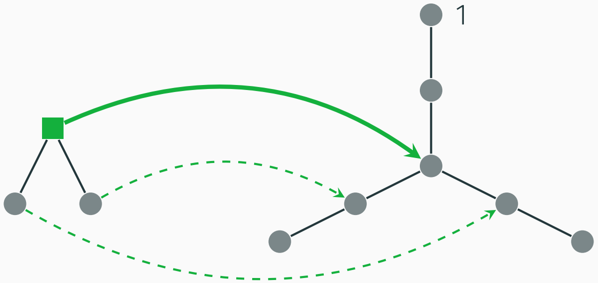
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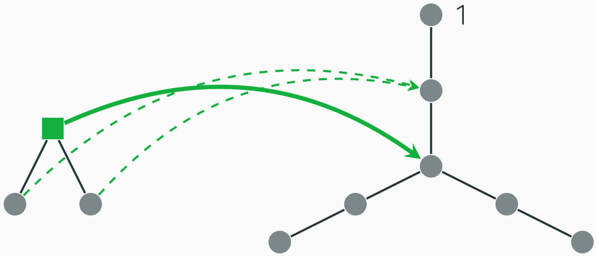
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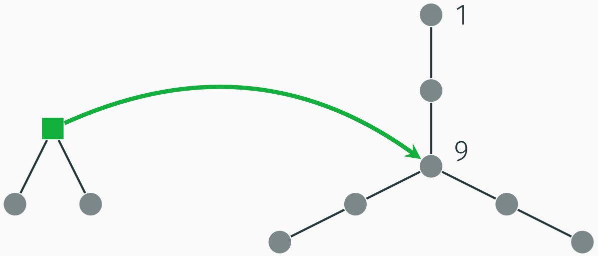
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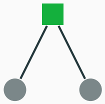
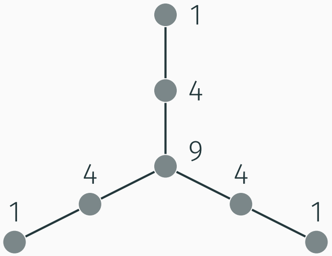
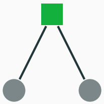
labelled graphs $\rightarrow$ homomorphism vectors $\subseteq \mathbb{R}^{V(G)}$



$\mapsto$





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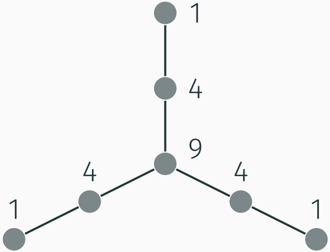

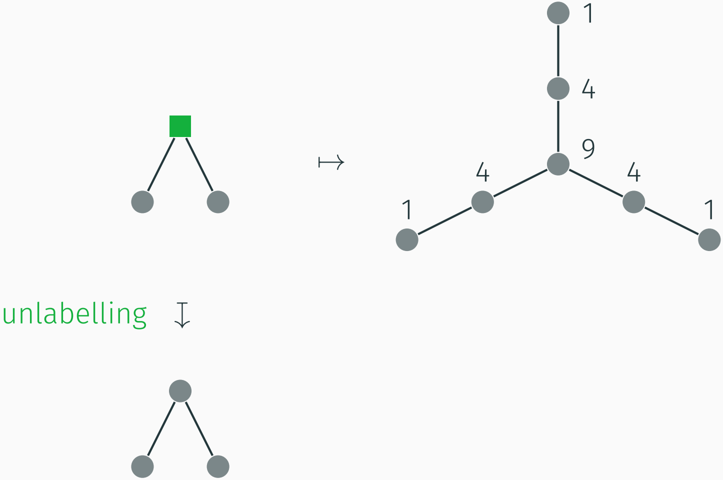
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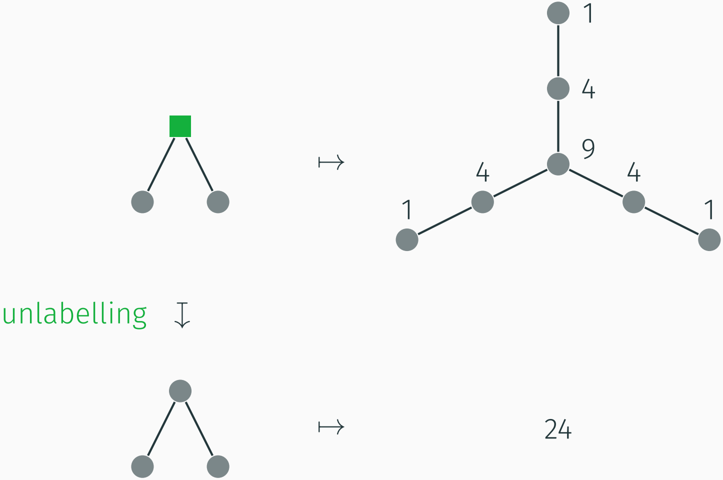
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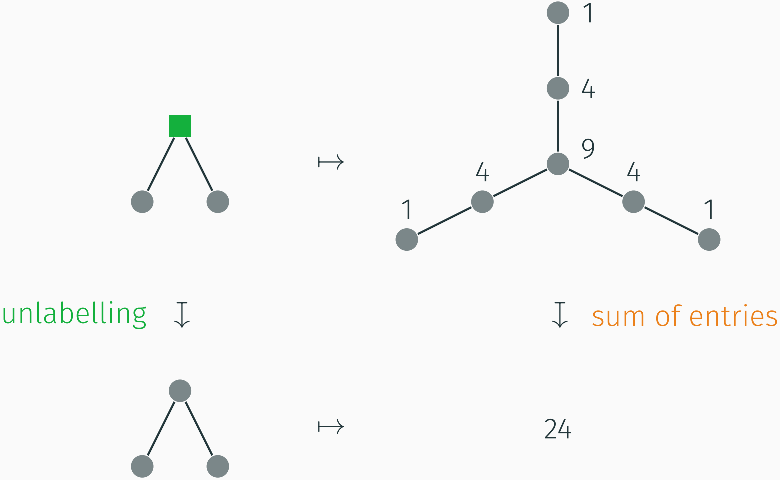

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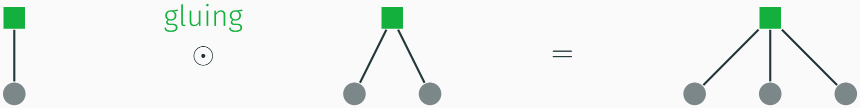
1
unlabelling ↧
↧ sum of entries
↦
24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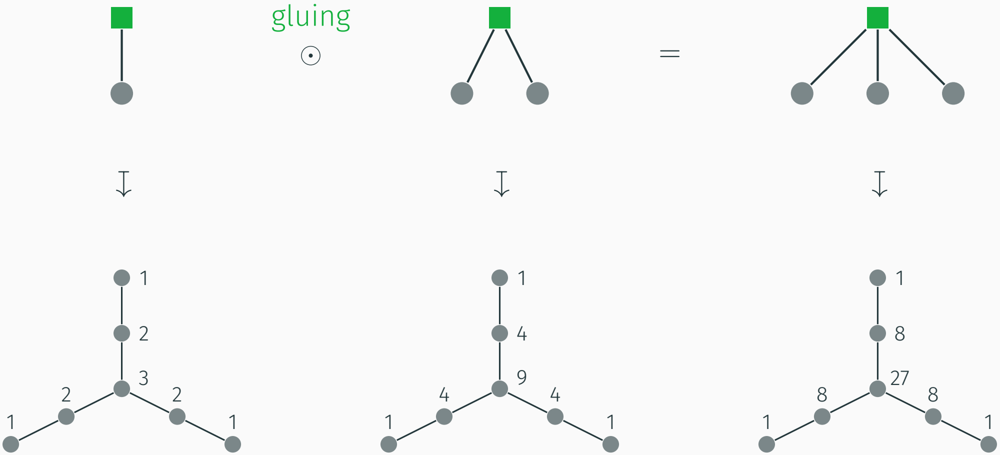
gluing
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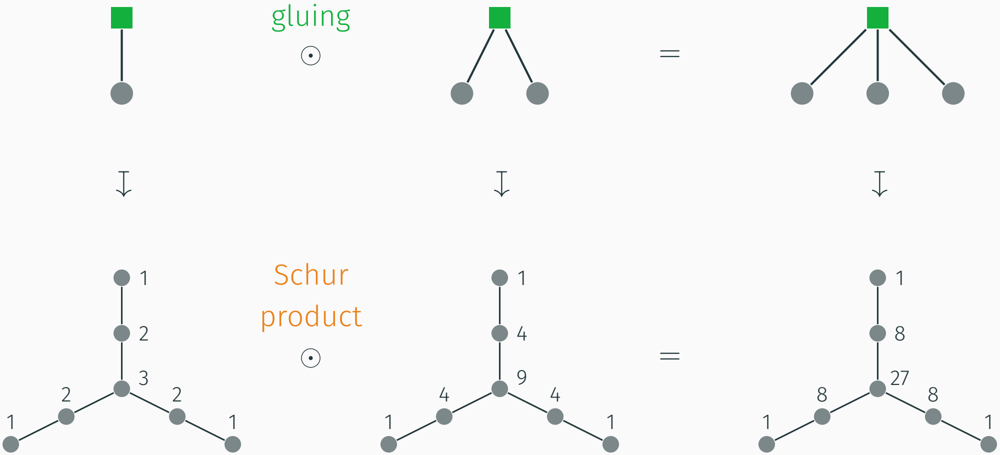
gluing
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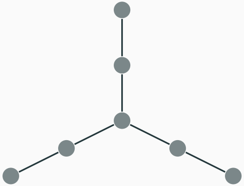
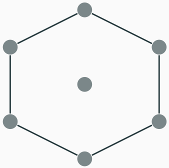
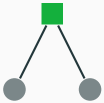
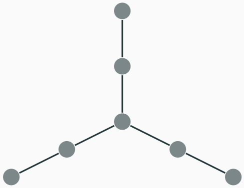


X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$

positive semi-definite







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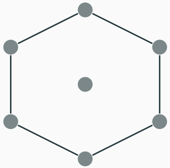
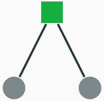
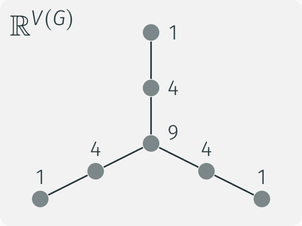
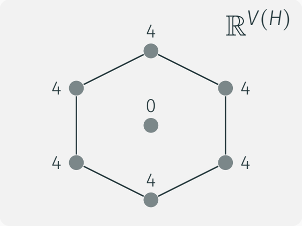


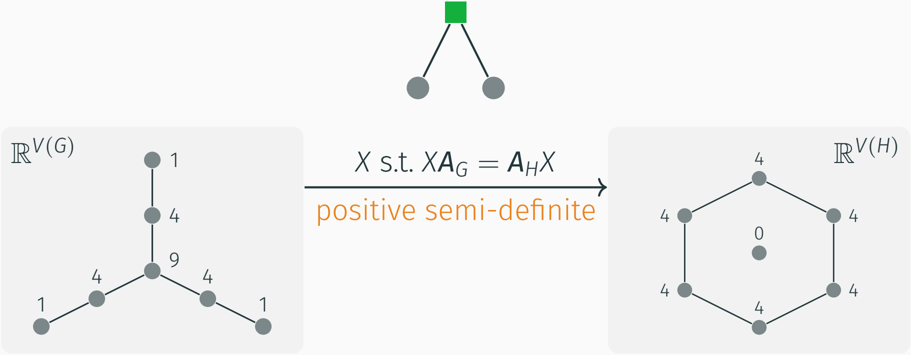


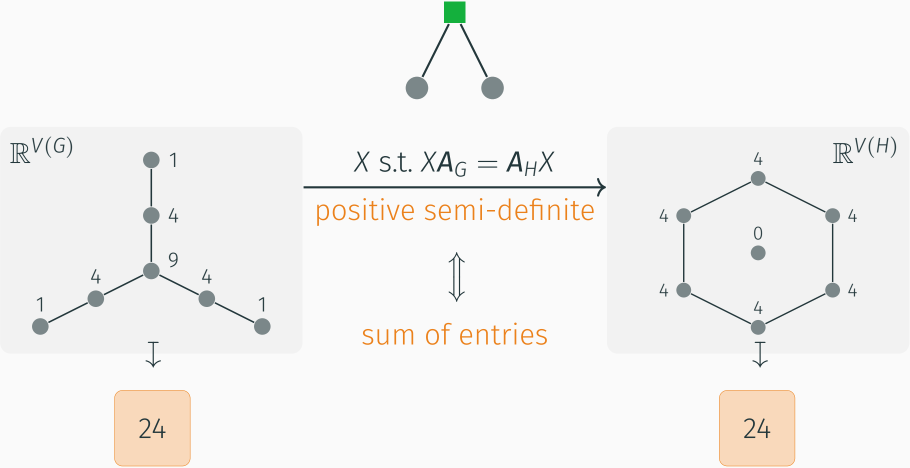


X s.t. $XA_G = A_H X$

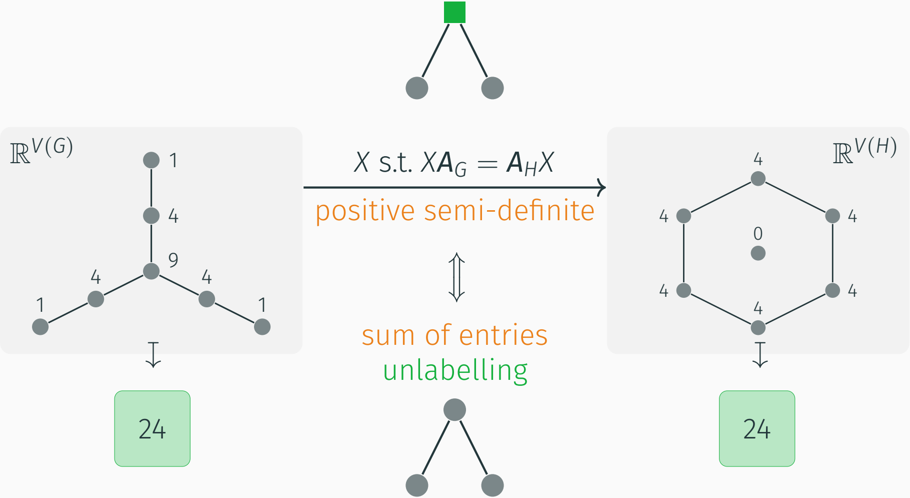
positive semi-definite



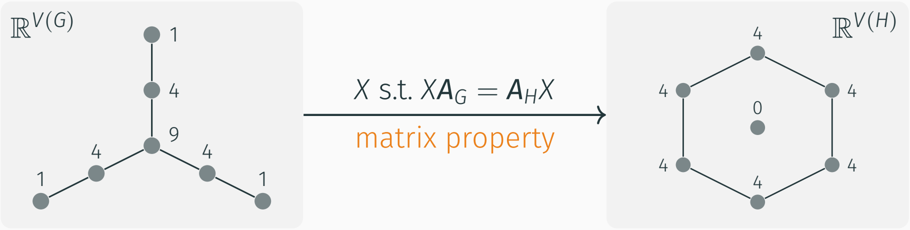

$\mathbb{R}^{V(G)}$
1
4
9
4
4
1
1
X s.t. $XA_G = A_HX$
positive semi-definite
$\mathbb{R}^{V(H)}$
4
4
4
0
4
4
4

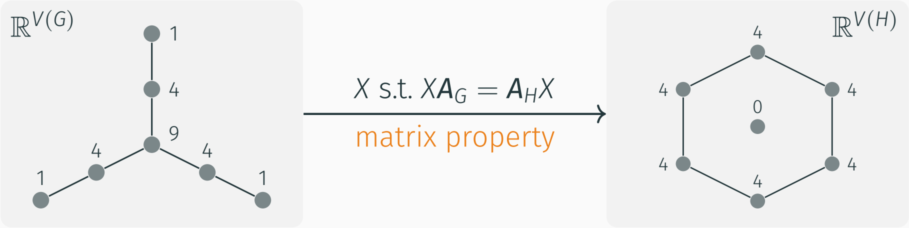


spaces of homomorphism vectors closed under algebraic operations

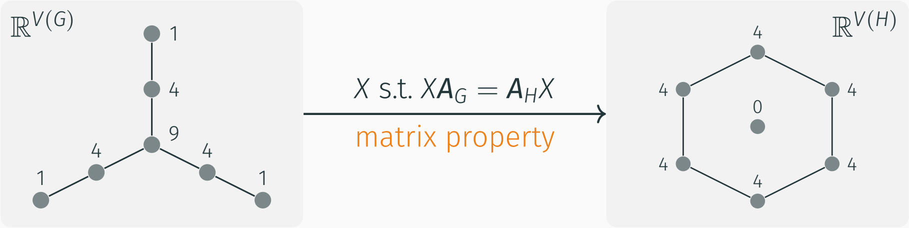


spaces of homomorphism vectors closed under algebraic operations
set of labelled graphs closed under combinatorial operations

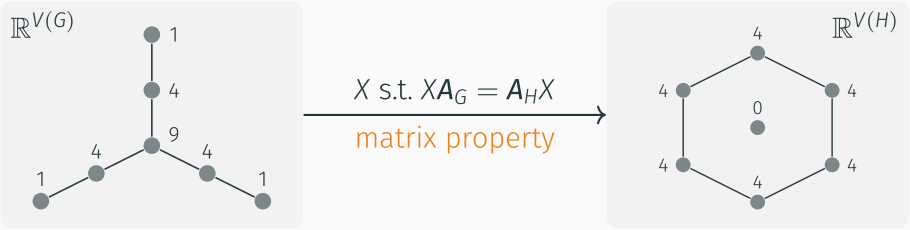

$\mathbb{R}^{V(G)}$
1
4
9
4
4
1
1
X s.t. $XA_G = A_H X$
matrix property
$\mathbb{R}^{V(H)}$
4
4
4
0
4
4
4



matrix property	graph class	
orthogonal	cycles	Specht (1940); Wiegmann (1961)



matrix property	graph class	
orthogonal	cycles	Specht (1940); Wiegmann (1961)
pseudo-stochastic	paths	**Grohe, Rattan, S. (2022)**
doubly stochastic	trees	**Grohe, Rattan, S. (2022)**



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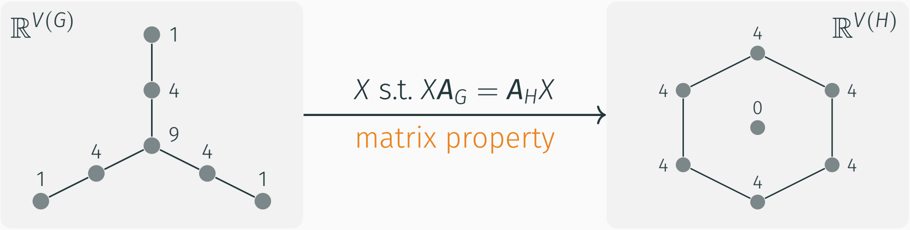
doubly stochastic	trees	**Grohe, Rattan, S. (2022)**
positive semi-definite	$\mathcal{L}_k$	Mančinska, Roberson, & Varvitsiotis (2023)



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positive semi-definite	$\mathcal{L}_k$	Mančinska, Roberson, & Varvitsiotis (2023)
quantum permutation	planar	Mančinska & Roberson (2020)
easy quantum orthog.		**S. and Spitzer (2024+)**

Equations

homomorphism vectors

algebraic operations

↔

Graph Class

labelled graphs

combinatorial operations

Equations

homomorphism vectors

algebraic operations

↔

Graph Class

labelled graphs

combinatorial operations

⋮

3
2
1

Lasserre

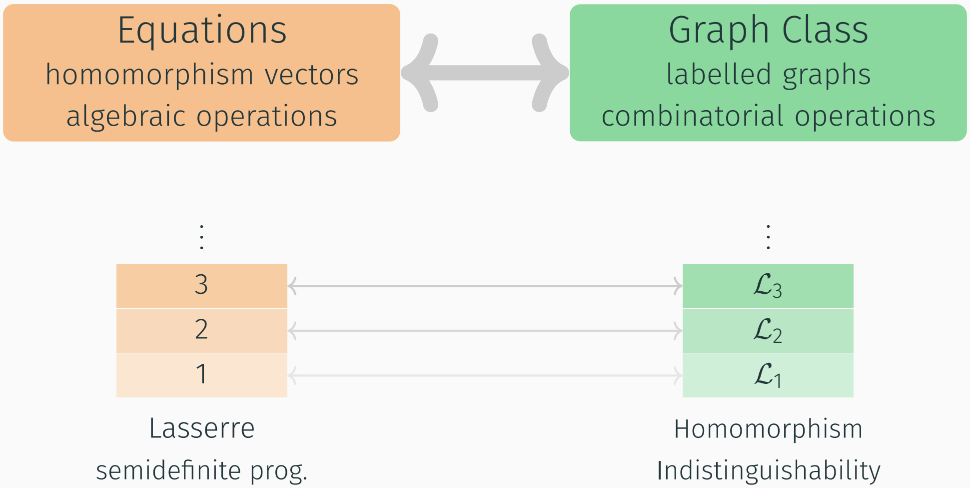
semidefinite prog.

Equations		Graph Class
homomorphism vectors	↔	labelled graphs
algebraic operations		combinatorial operations

Lasserre semidefinite prog.	Homomorphism Indistinguishability
⋮	⋮
3	$\mathcal{L}_3$
2	$\mathcal{L}_2$
1	$\mathcal{L}_1$

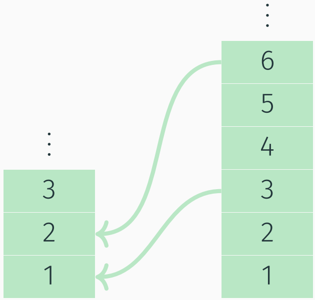
Lasserre
semidefinite prog.

Homomorphism
Indistinguishability


Equations
homomorphism vectors
algebraic operations
Graph Class
labelled graphs
combinatorial operations
3
2
1
$\mathcal{L}_3$
$\mathcal{L}_2$
$\mathcal{L}_1$
Lasserre
semidefinite prog.
Homomorphism
Indistinguishability

Characterisations

How to characterise $\equiv_{\mathcal{F}}$?



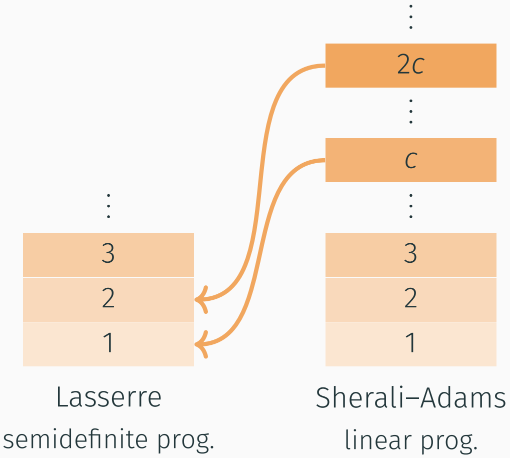
Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?

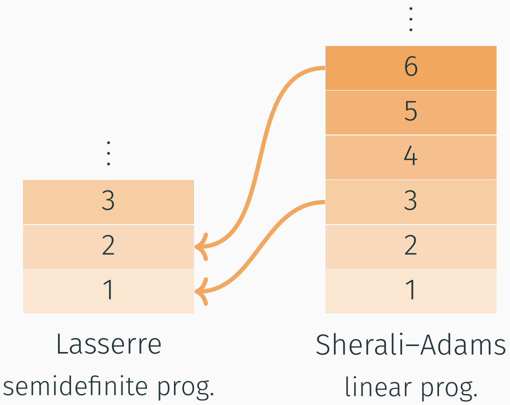
Complexity

How to test $\equiv_{\mathcal{F}}$?

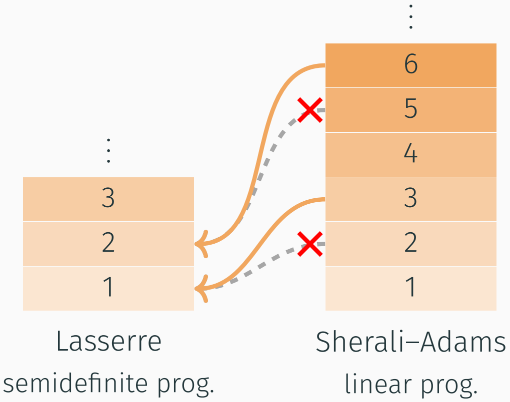
Lasserre	Sherali–Adams
⋮	⋮
3	3
2	2
1	1
Lasserre	Sherali–Adams
semidefinite prog.	linear prog.



Atserias & Ochremiak (2018)



Atserias & Ochremiak (2018), **Roberson and S. (2023)**



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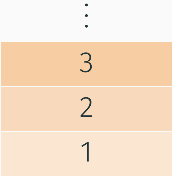
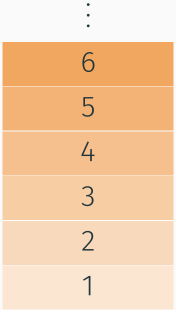


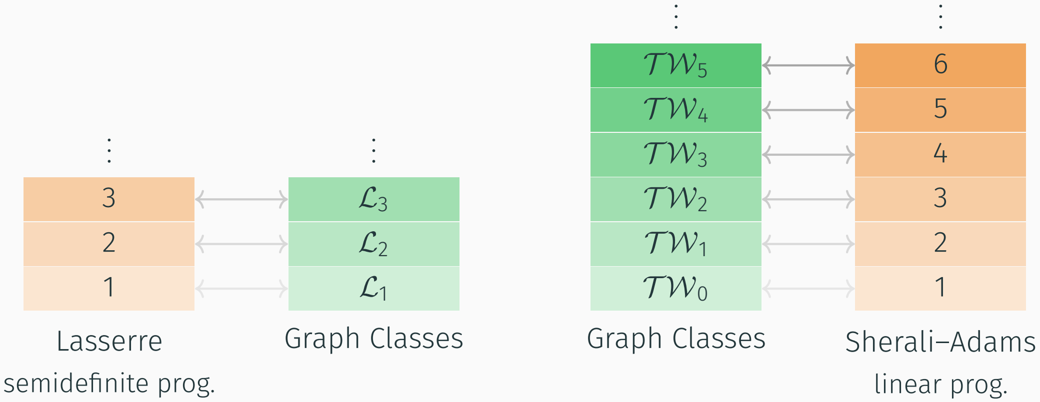
Lasserre

semidefinite prog.



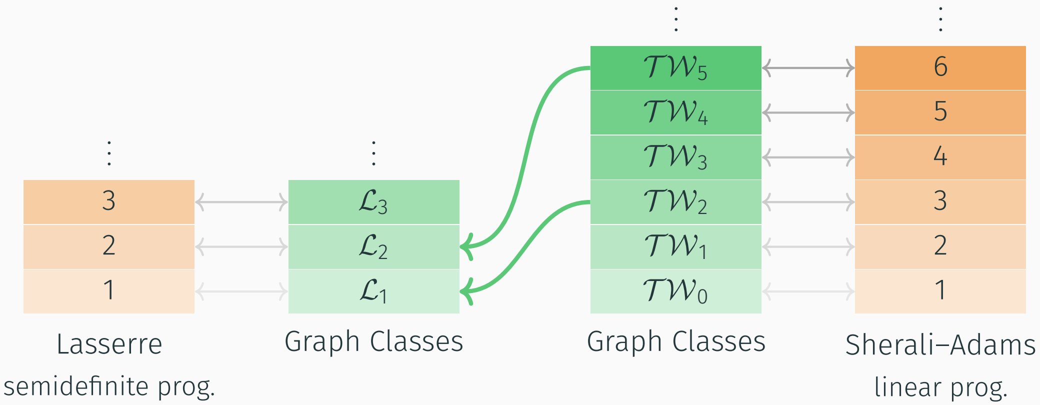
Sherali–Adams

linear prog.



Roberson and S. (2023); Atserias & Maneva (2012); Malkin (2014); Dvořák (2010); Dell, Grohe, & Rattan (2018)

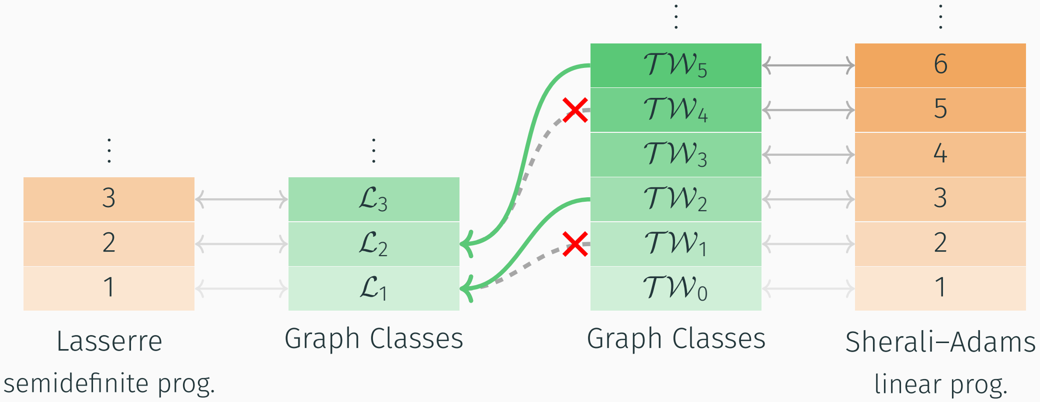
$\mathcal{L}_k \subseteq \mathcal{TW}_{3k-1}$



Roberson and S. (2023); Atserias & Maneva (2012); Malkin (2014); Dvořák (2010); Dell, Grohe, & Rattan (2018)

$\mathcal{L}_k \subseteq \mathcal{TW}_{3k-1}$

$\mathcal{L}_k$ contains a graph of treewidth $3k-1$



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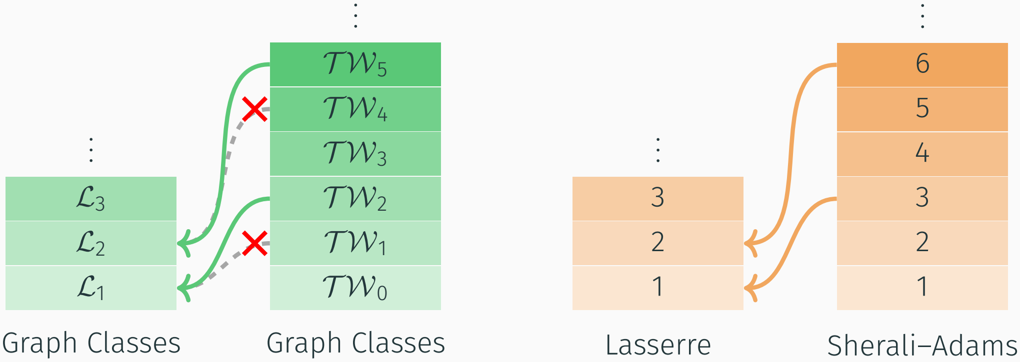
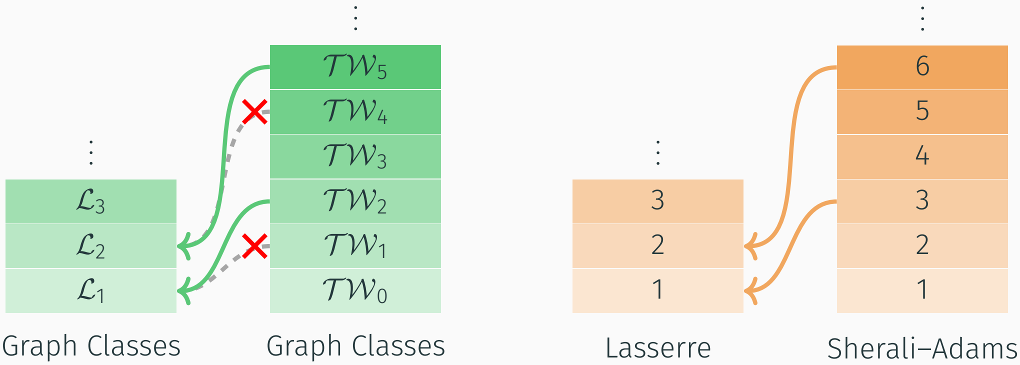

$\mathcal{L}_3$
$\mathcal{L}_2$
$\mathcal{L}_1$
Graph Classes
$\mathcal{TW}_5$
$\mathcal{TW}_4$
$\mathcal{TW}_3$
$\mathcal{TW}_2$
$\mathcal{TW}_1$
$\mathcal{TW}_0$
Graph Classes
3
2
1
Lasserre
6
5
4
3
2
1
Sherali–Adams

Definition (Roberson (2022))

A graph class $\mathcal{F}$ is homomorphism distinguishing closed if for all graph classes $\mathcal{K}$

$$\mathcal{K} \text{ is contained in } \mathcal{F} \iff \equiv_{\mathcal{F}} \text{ refines } \equiv_{\mathcal{K}}.$$



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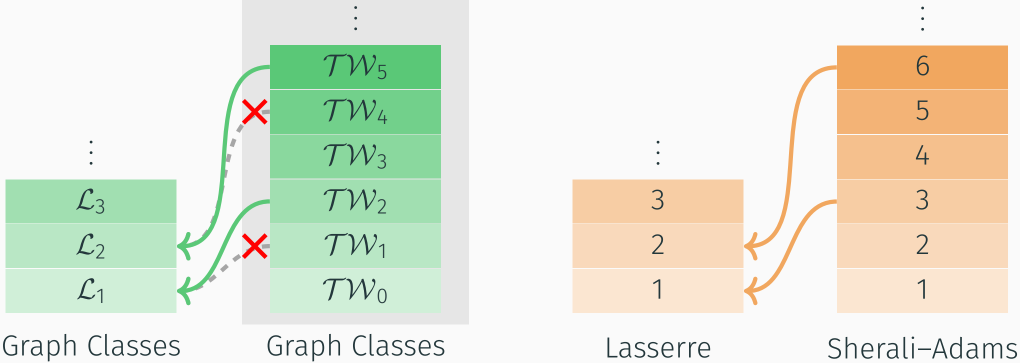

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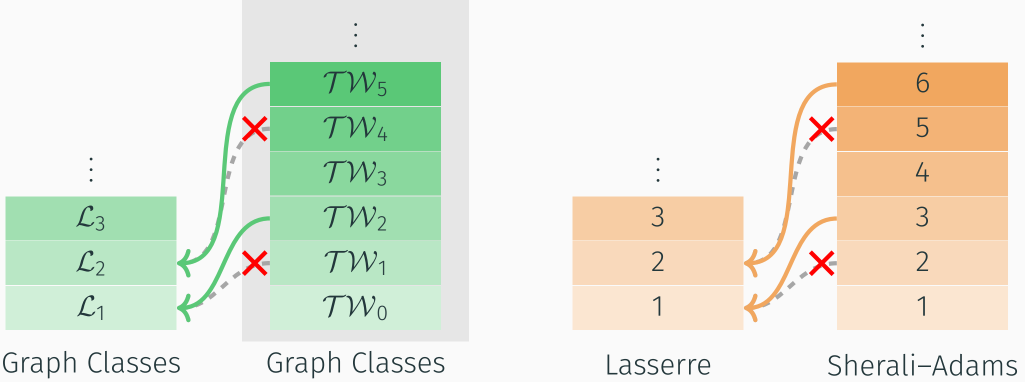
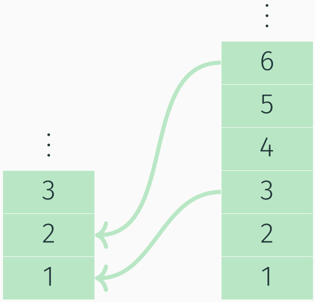
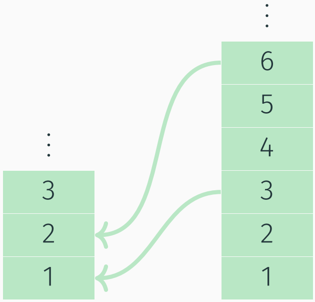




Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?



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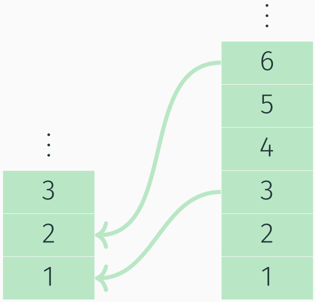
- compared distinguishing power of Lasserre and Sherali–Adams by comparing $\mathcal{TW}_k$ and $\mathcal{L}_k$



Distinguishing Power

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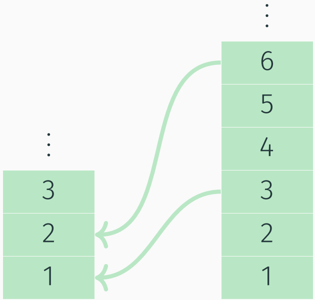
- compared distinguishing power of Lasserre and Sherali–Adams by comparing $\mathcal{TW}_k$ and $\mathcal{L}_k$
- distinguishing power of $\equiv_{\mathcal{TW}_k}$ is described by fact that $\mathcal{TW}_k$ is homomorphism distinguishing closed



Distinguishing Power

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Theory of Homomorphism Indistinguishability

When is a graph class $\mathcal{F}$ homomorphism distinguishing closed?

Conjecture (Roberson (2022))

Every minor-closed union-closed graph class is homomorphism distinguishing closed.

Conjecture (Roberson (2022))

Every minor-closed union-closed graph class is homomorphism distinguishing closed.

- planar graphs — Roberson (2022)
- bounded treewidth — Neuen (2024)
- bounded treedepth — **Fluck, S., & Spitzer (2024)**
- bounded pathwidth — **S. (2024)**
- essentially finite graph classes — **S. (2023)**
- outerplanar graphs — **Neuen & S. (2024)**

Theorem (S. (2023))

For every homomorphism distinguishing closed graph class $\mathcal{F}$,

$$\mathcal{F} \text{ is minor-closed} \iff \equiv_{\mathcal{F}} \text{ is preserved under complements.}$$

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- Typical graph isomorphism relaxations are preserved under complements.

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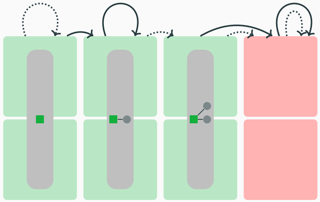
- Typical graph isomorphism relaxations are preserved under complements.
- Towards a **theory of homomorphism indistinguishability**, we can focus on minor-closed graph classes.
- Minor-closed graph classes are subject to a rich structure theory.

Characterisations

How to characterise $\equiv_{\mathcal{F}}$?

Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?



Complexity

How to test $\equiv_{\mathcal{F}}$?

Let $\mathcal{F}$ be minor-closed and proper.

HomInd($\mathcal{F}$)	
Input	Graphs G and H.
Decide	$G \equiv_{\mathcal{F}} H$.

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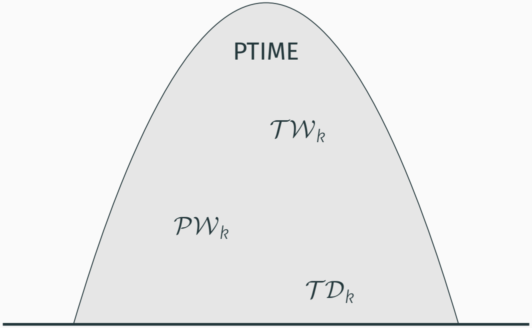

Dell, Grohe, & Rattan (2018); Dvořák (2010); Grohe (2020); **Grohe, Rattan, S. (2022)**

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$\textsc{HomInd}(\mathcal{F})$	
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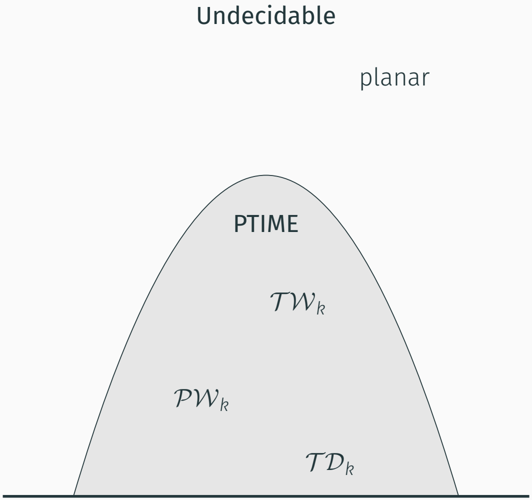
Decide	$G \equiv_{\mathcal{F}} H$.



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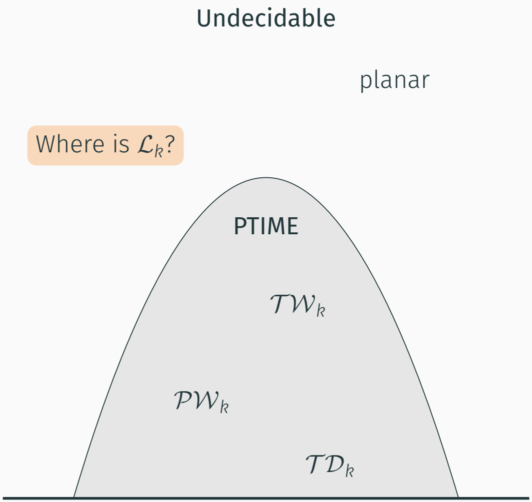
HOMIND($\mathcal{F}$)	
Input	Graphs G and H.
Decide	$G \equiv_{\mathcal{F}} H$.



Dell, Grohe, & Rattan (2018); Dvořák (2010); Grohe (2020); **Grohe, Rattan, S. (2022)**; Mančinska & Roberson (2020)

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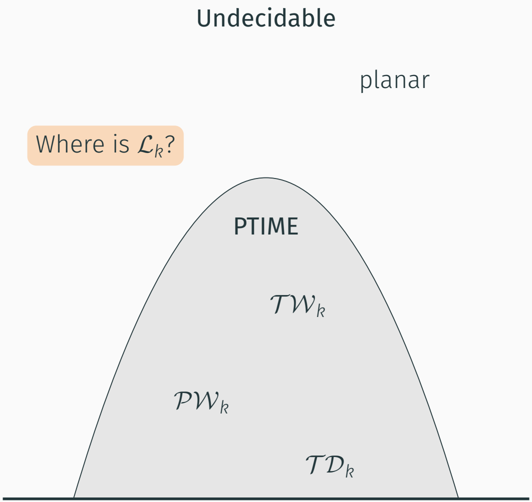
HomInd($\mathcal{F}$)

Input Graphs G and H.

Decide $G \equiv_{\mathcal{F}} H$.

Theorem (S. (2024))

If $\mathcal{F}$ has bounded treewidth, then HomInd($\mathcal{F}$) *is in* coRP.



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Input Graphs G and H.

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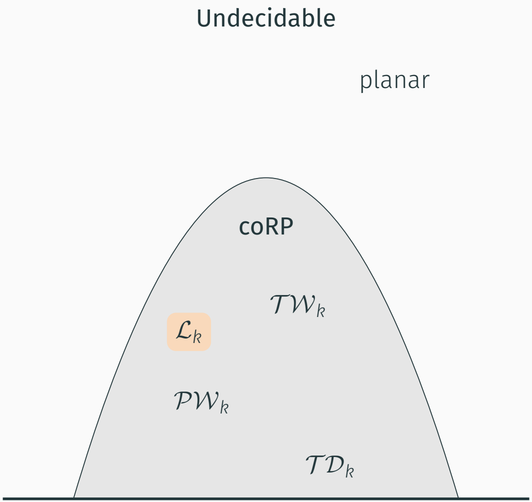

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Theorem (S. (2024))

For every minor-closed graph class $\mathcal{F}$ of bounded treewidth, $\textsc{HomInd}(\mathcal{F})$ *is in* coRP.

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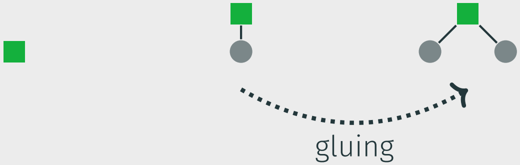

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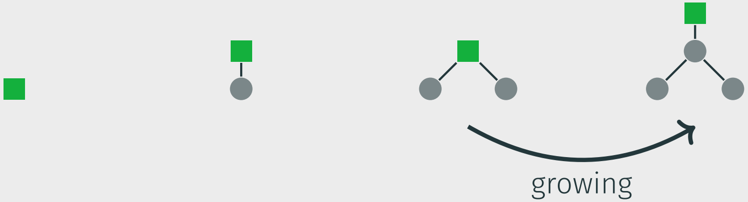
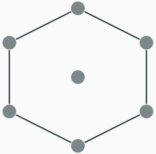
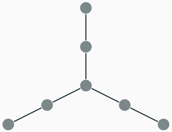
For every minor-closed graph class $\mathcal{F}$ of bounded treewidth, HomInd($\mathcal{F}$) *is in* coRP.

$\mathcal{F} =$ trees

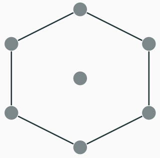




Space of homomorphism vectors of labelled trees



Space of homomorphism vectors of labelled trees



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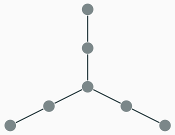

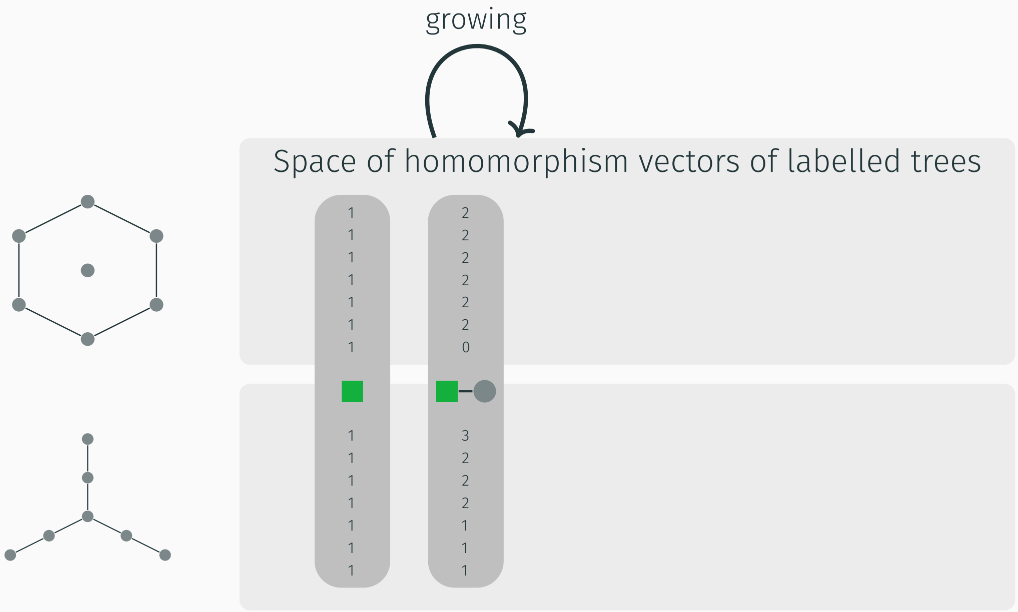
growing
Space of homomorphism vectors of labelled trees
1 1 1 1 1 1 1
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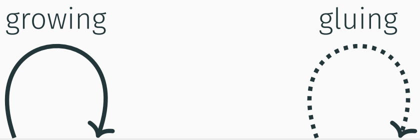
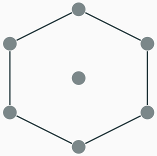
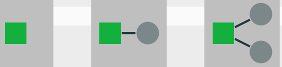
1 1 1 1 1 1 1
3 2 2 2 1 1 1

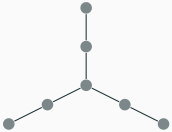


Space of homomorphism vectors of labelled trees

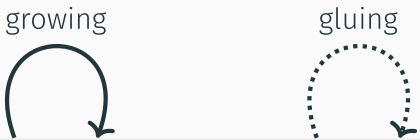


1	2	4
1	2	4
1	2	4
1	2	4
1	2	4
1	2	4
1	0	0

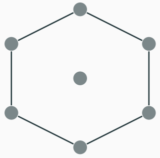




1	3	9
1	2	4
1	2	4
1	2	4
1	1	1
1	1	1
1	1	1



Space of homomorphism vectors of labelled trees



1	2	4	8	32
1	2	4	8	32
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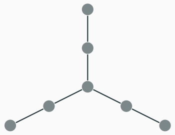
1	2	4	8	32
1	2	4	8	32
1	2	4	8	32
1	0	0	0	0



1	3	9	12	108
1	2	4	10	40
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1	2	4	10	40
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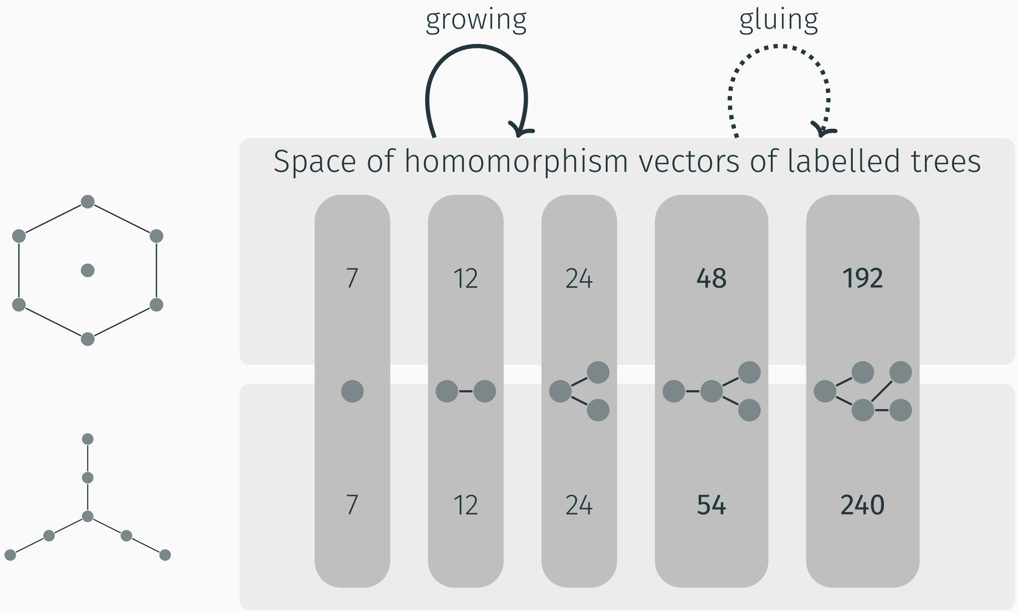
1	1	1	4	4
1	1	1	4	4


growing
gluing
Space of homomorphism vectors of labelled trees
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192
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Theorem (S. (2024))

For every minor-closed graph class $\mathcal{F}$ of bounded treewidth, $\textsc{HomInd}(\mathcal{F})$ *is in* coRP.

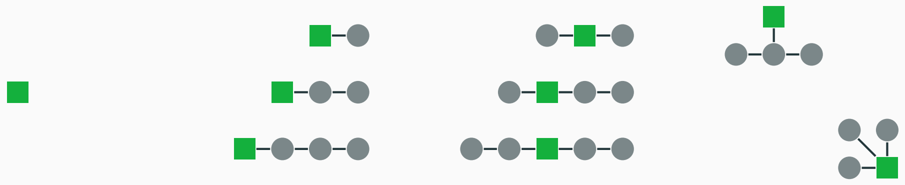
$\mathcal{F} =$ trees

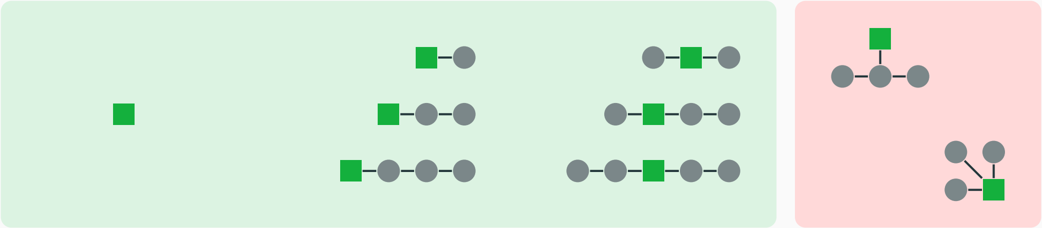
Theorem (S. (2024))

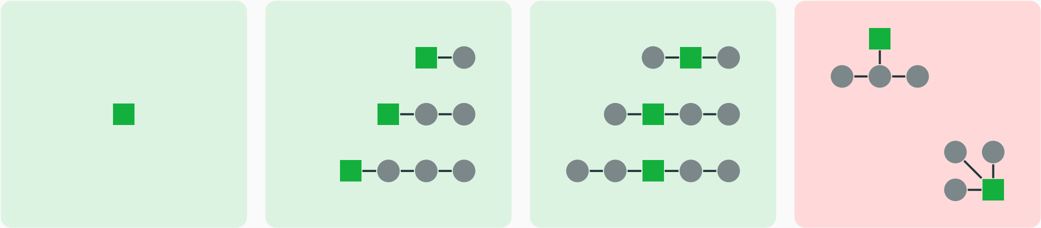
For every minor-closed graph class $\mathcal{F}$ of bounded treewidth, HomInd($\mathcal{F}$) *is in* coRP.

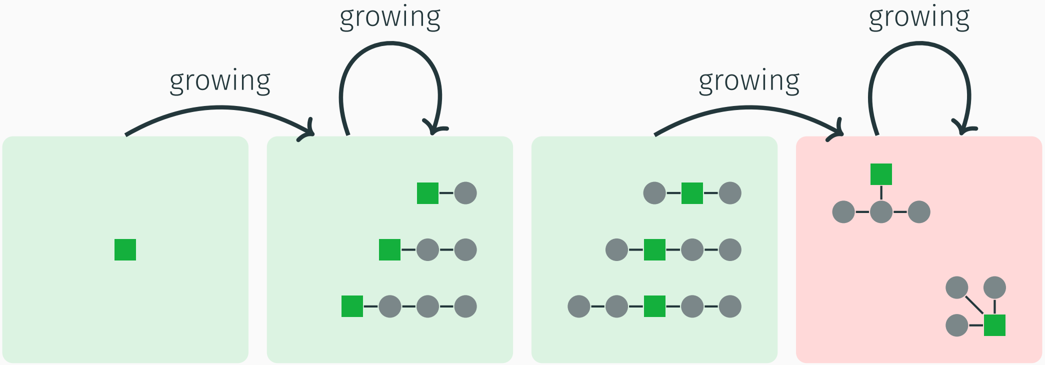
Trees

$\mathcal{F}$ = paths








growing
growing
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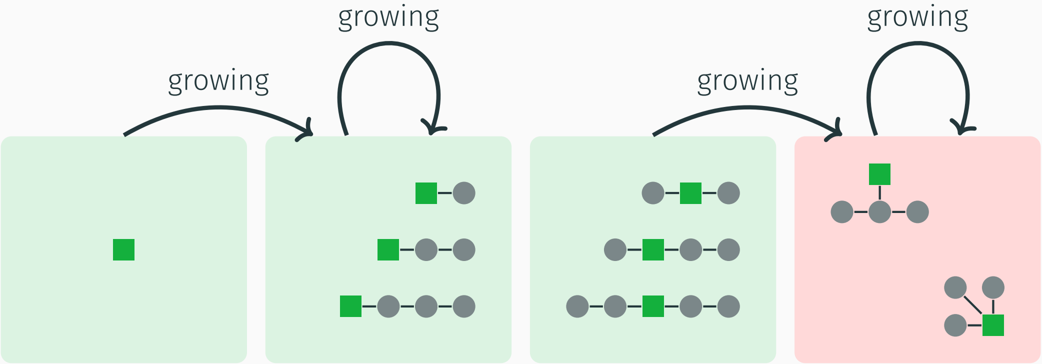
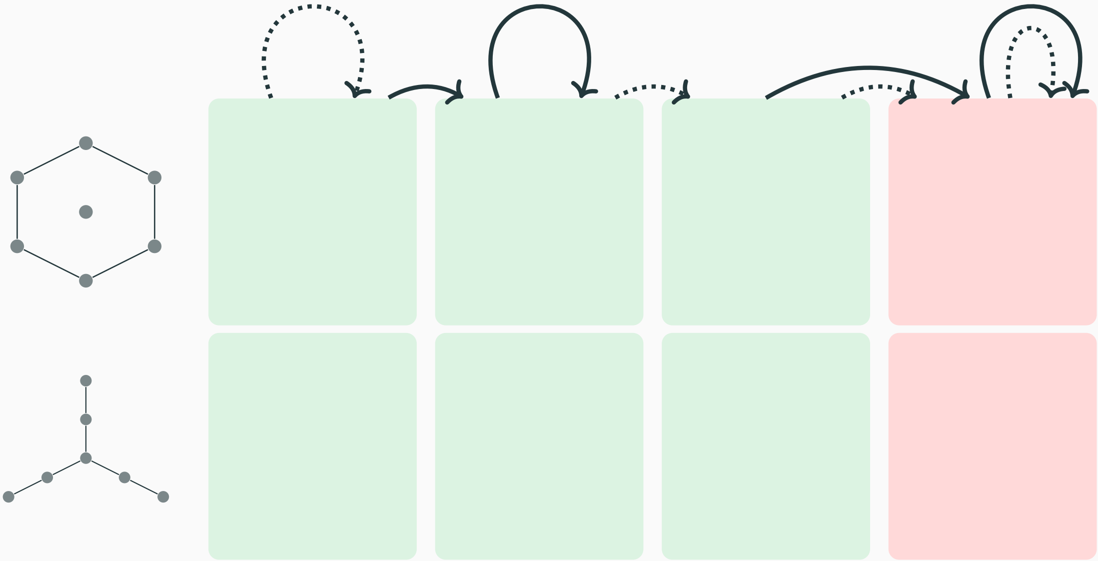
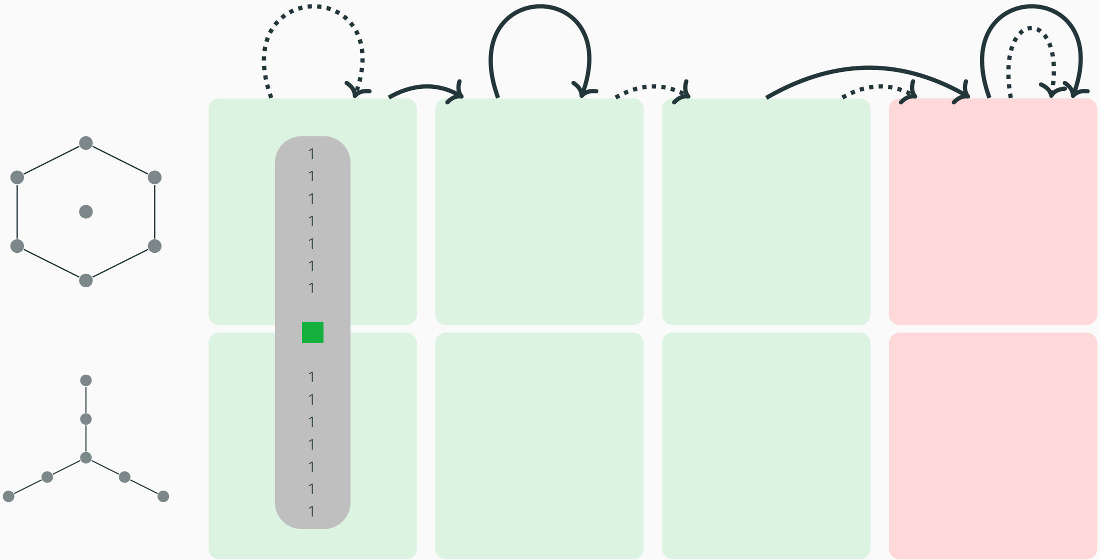


Theorem (Courcelle (1990); Robertson & Seymour (2004))

Every minor-closed graph class $\mathcal{F}$ induces finitely many classes.

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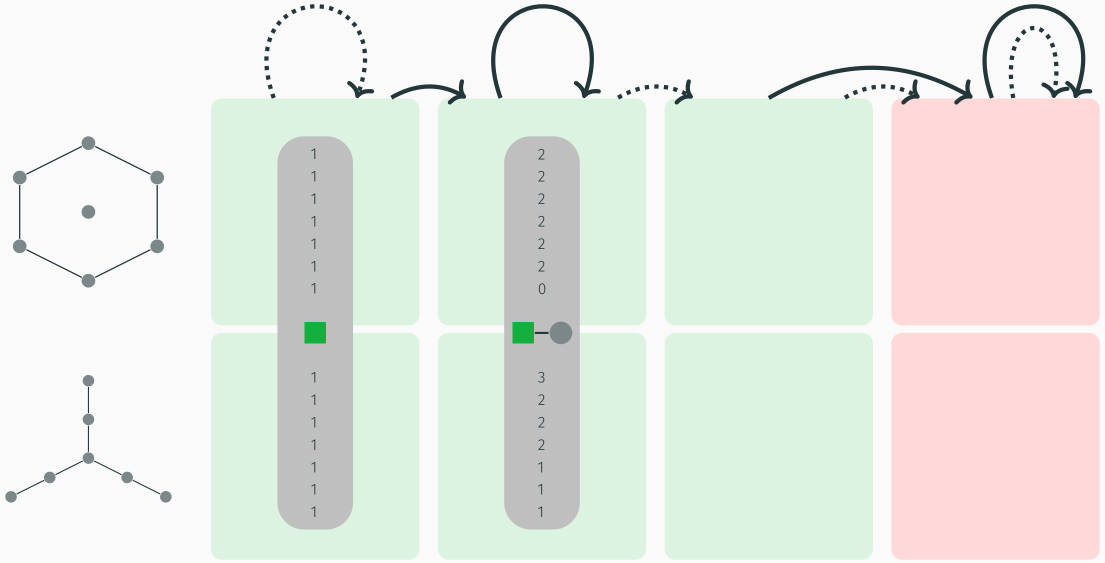
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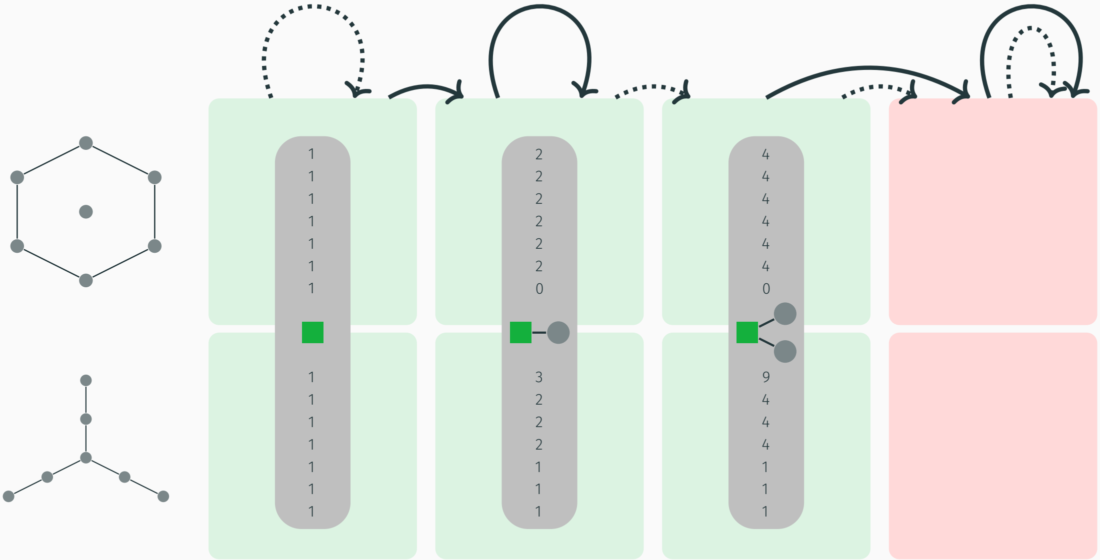
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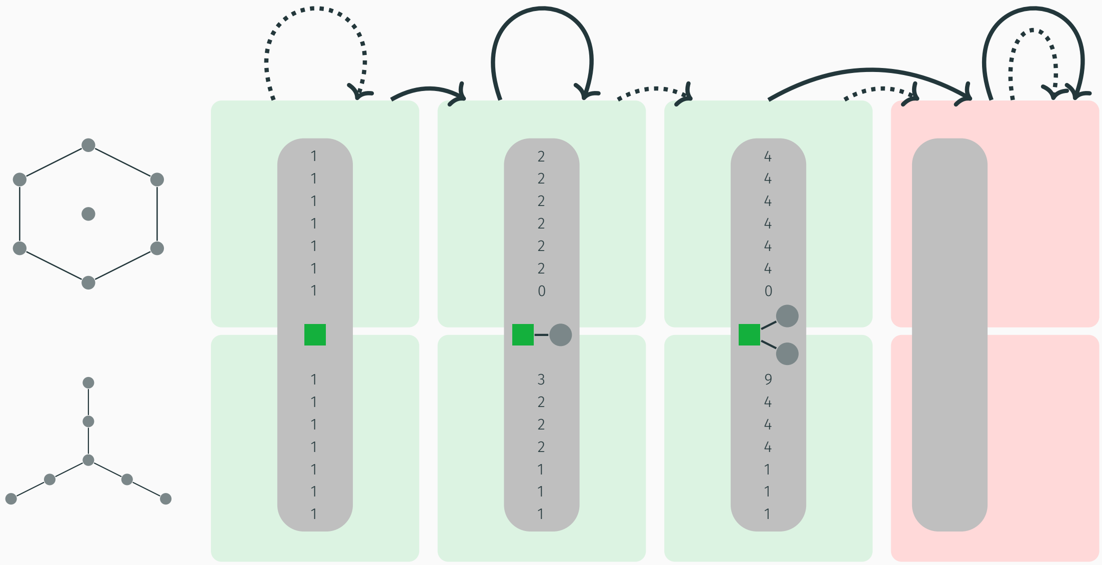
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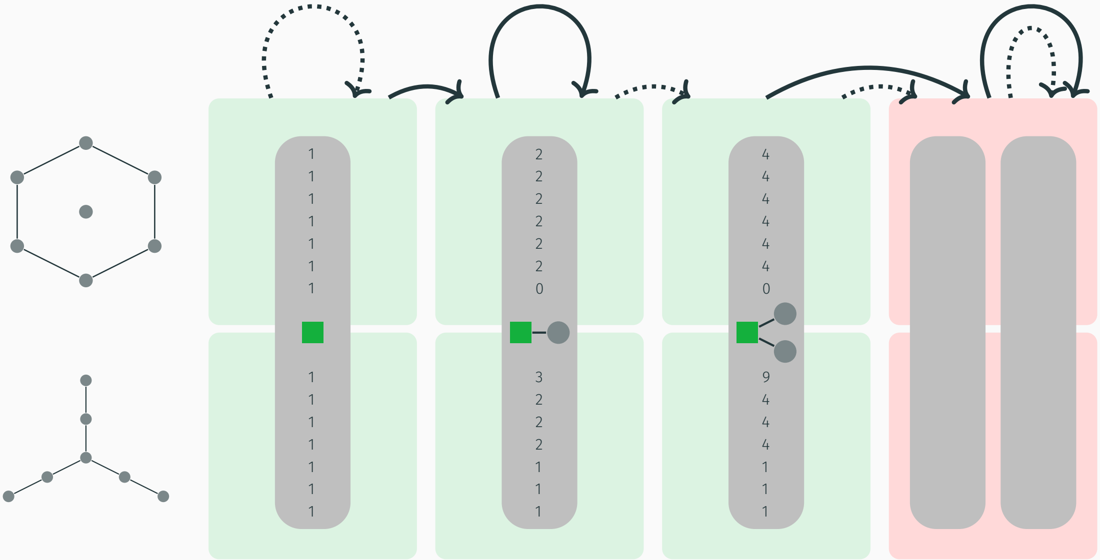
1 1 1 1 1 1 1
2 2 2 2 2 2 0
3 2 2 2 1 1 1
4 4 4 4 4 4 0
9 4 4 4 1 1 1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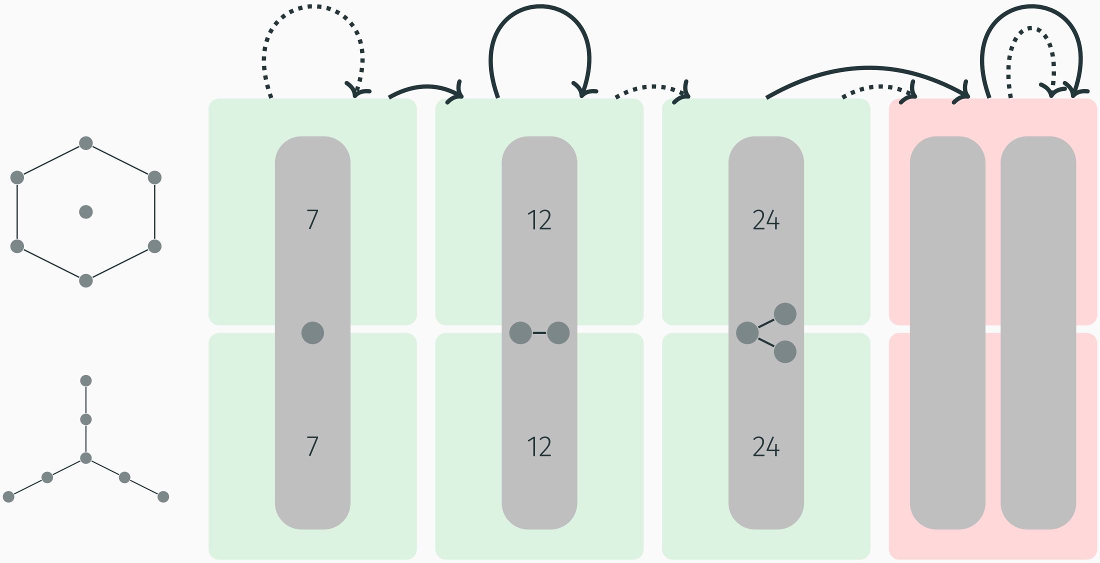
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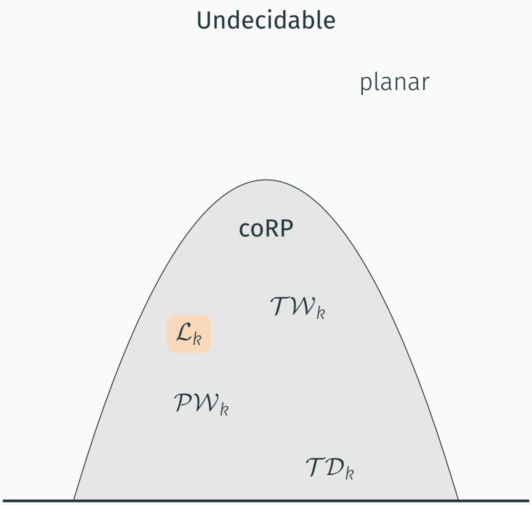
Let $\mathcal{F}$ be minor-closed and proper.

Theorem (S. (2024))

If $\mathcal{F}$ has bounded treewidth, then HomInd($\mathcal{F}$) *is in* coRP.



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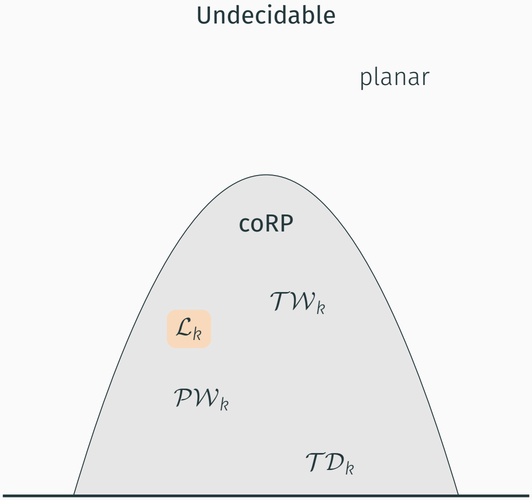
Conjecture (S. (2024))

If $\mathcal{F}$ has bounded treewidth, then HomInd($\mathcal{F}$) *is in* PTIME.



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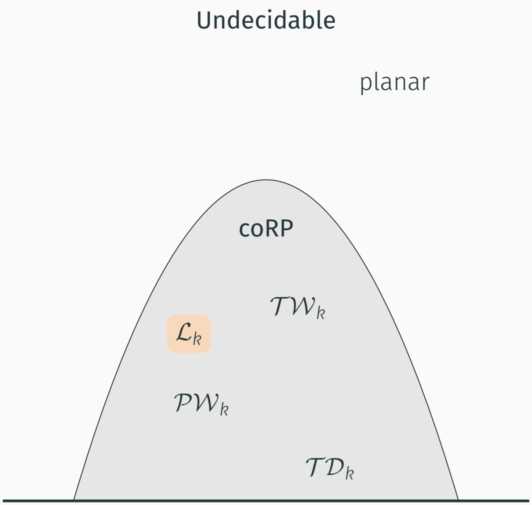
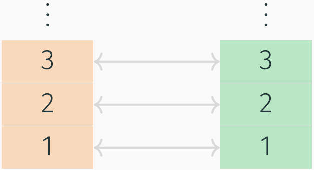
Conjecture (S. (2024))

If $\mathcal{F}$ has bounded treewidth, then HomInd($\mathcal{F}$) *is in* PTIME.

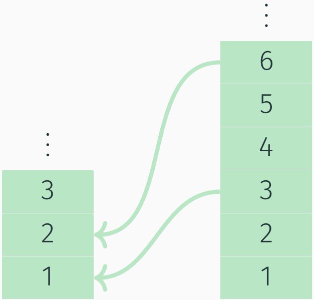
Otherwise, HomInd($\mathcal{F}$) *is undecidable.*





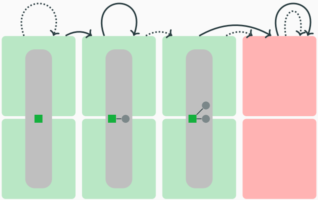
Characterisations

How to characterise $\equiv_{\mathcal{F}}$?



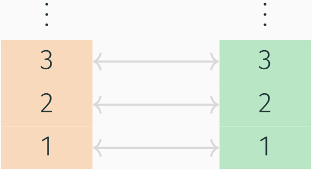
Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?



Complexity

How to test $\equiv_{\mathcal{F}}$?



Characterisations

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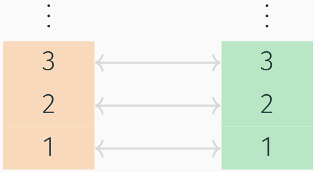
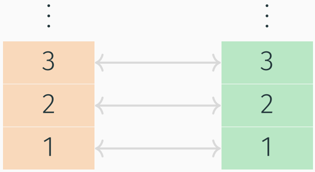


Characterisations

How to characterise $\equiv_{\mathcal{F}}$?

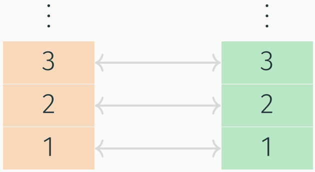
- **Tools:** labelled graphs and homomorphism vectors



Characterisations

How to characterise $\equiv_{\mathcal{F}}$?

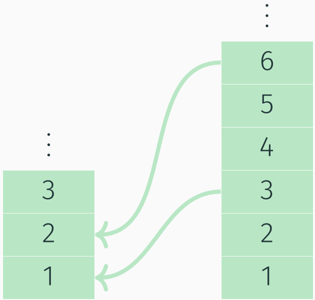
- **Tools:** labelled graphs and homomorphism vectors
- **Results:** variants of Specht–Wiegmann Theorem



Characterisations

How to characterise $\equiv_{\mathcal{F}}$?

- **Tools:** labelled graphs and homomorphism vectors
- **Results:** variants of Specht–Wiegmann Theorem
- **Lasserre** is a homomorphism indistinguishability relation.



Distinguishing Power

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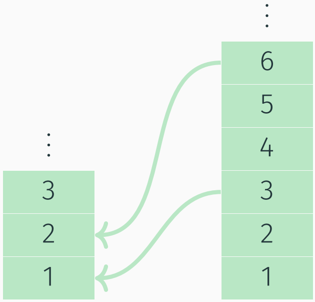
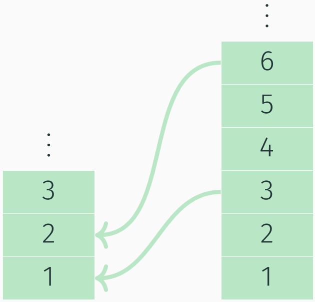


Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?

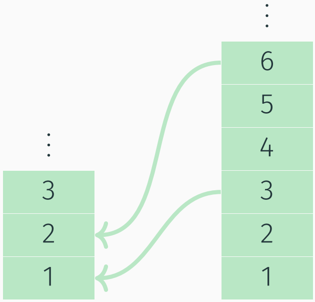
- Comparing graph isomorphism relaxations by comparing graph classes



Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?

- Comparing graph isomorphism relaxations by comparing graph classes
- Determined power of **Lasserre** vis-à-vis Sherali–Adams



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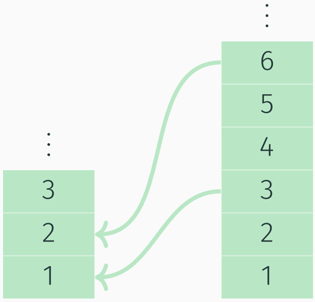

Distinguishing Power

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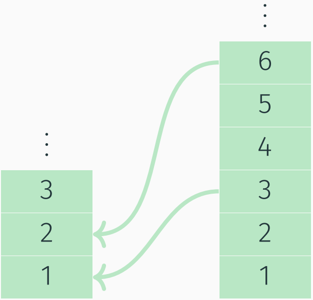
- Determined power of **Lasserre** vis-à-vis Sherali–Adams

Theory of Homomorphism Indistinguishability



Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?

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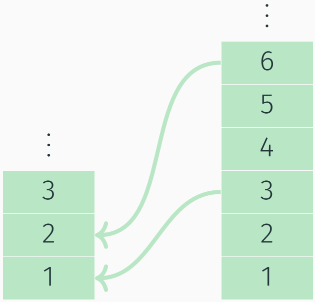
- Determined power of **Lasserre** vis-à-vis Sherali–Adams

Theory of Homomorphism Indistinguishability

- **Result:** minor-closed graph classes play a central role.



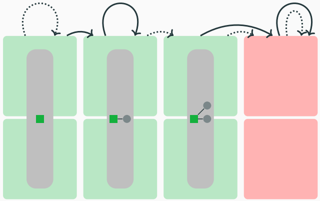
Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?

- Comparing graph isomorphism relaxations by comparing graph classes
- Determined power of **Lasserre** vis-à-vis Sherali–Adams

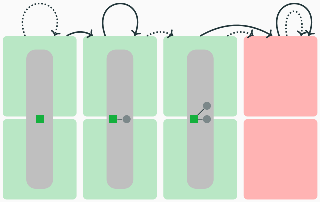
Theory of Homomorphism Indistinguishability

- **Result:** minor-closed graph classes play a central role.
- **Open:** Roberson's conjecture



Complexity

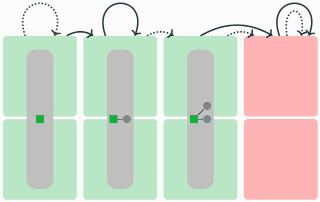
How to decide $\equiv_{\mathcal{F}}$?



Complexity

How to decide $\equiv_{\mathcal{F}}$?

Theory of Homomorphism Indistinguishability

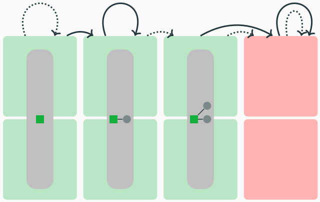


Complexity

How to decide $\equiv_{\mathcal{F}}$?

Theory of Homomorphism Indistinguishability

- **Result:** HomInd($\mathcal{F}$) is in coRP for minor-closed graph classes $\mathcal{F}$ of bounded treewidth.

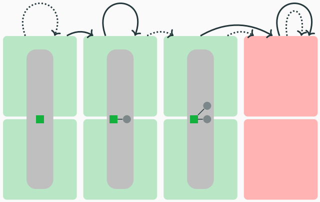


Complexity

How to decide $\equiv_{\mathcal{F}}$?

Theory of Homomorphism Indistinguishability

- **Result:** $\textsc{HomInd}(\mathcal{F})$ is in coRP for minor-closed graph classes $\mathcal{F}$ of bounded treewidth.
- **Open:** dichotomy for proper minor-closed graph classes



Complexity

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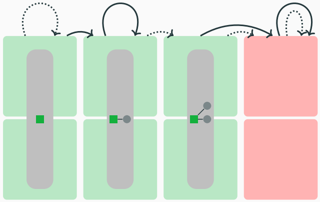
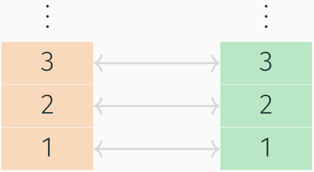


Complexity
How to decide $\equiv_{\mathcal{F}}$?

Theory of Homomorphism Indistinguishability

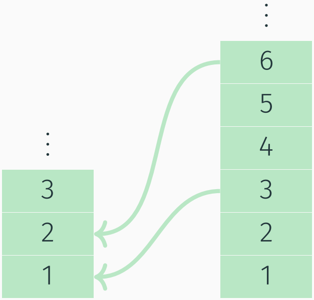
- **Result:** $\textsc{HomInd}(\mathcal{F})$ is in coRP for minor-closed graph classes $\mathcal{F}$ of bounded treewidth.
- **Open:** dichotomy for proper minor-closed graph classes

- **Lasserre** is in coRP.



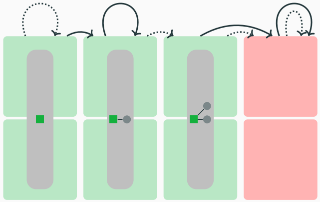
Characterisations

How to characterise $\equiv_{\mathcal{F}}$?



Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?



Complexity

How to test $\equiv_{\mathcal{F}}$?

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