Symmetric Circuits and Homomorphism Polynomials

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Joint work with Anuj Dawar and Benedikt Pago (University of Cambridge)

Algebraic circuits

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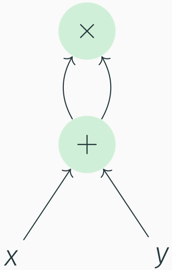

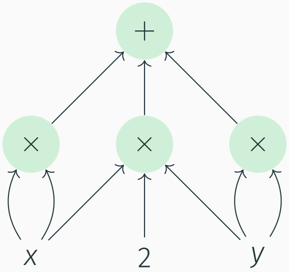

VP vs. VNP

Determinant

$$\det_n = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} x_{i,\pi(i)}$$

Permanent

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Theorem (Dawar & Wilsenach (2020))

det_n *admits poly-size symmetric circuits.*

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perm_n *does not admit poly-size symmetric circuits.*

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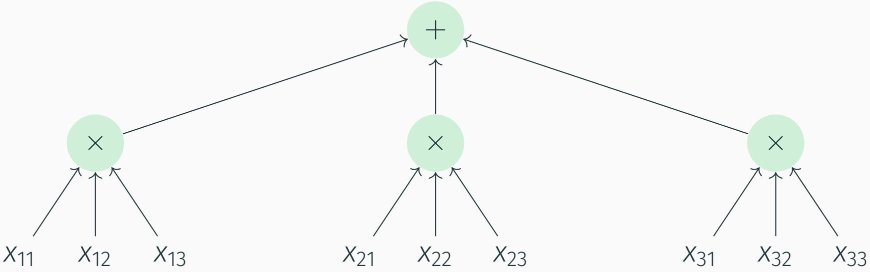

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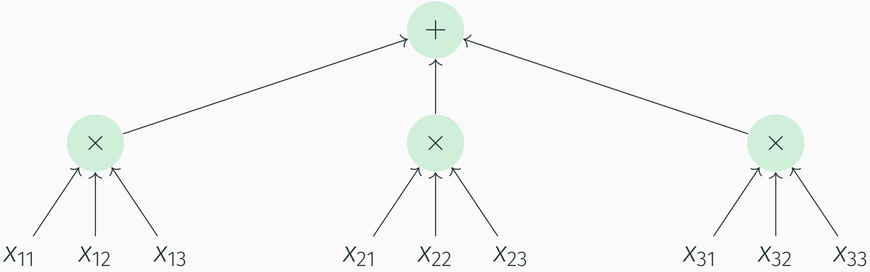

- Symmetric polynomials are functions of (n, n)-vertex bipartite graphs.
  E.g., the permanent $\mathrm{perm}_n(G)$ is the number of perfect matchings in G.

poly-size
symmetric circuits

poly-size symmetric circuits $\rightarrow$ symmetric circuits with poly-sized orbits

$(\mathrm{perm}_n)_{n\in\mathbb{N}} \notin$

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symmetric circuits → symmetric circuits
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polynomials of bounded-treewidth patterns

For a bipartite multigraph F and $n \in \mathbb{N}$,

$$\hom_{F,n} := \sum_{h\colon A \uplus B \to [n] \uplus [n]} \prod_{uv \in E(F)} x_{h(uv)}$$

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Fact

A polynomial is symmetric $\iff$ it is a linear combination of $\hom$-polynomials.

Theorem (Dawar, Pago, & S. (2024))

For a sequence of symmetric polynomials $(p_n)_{n \in \mathbb{N}}$, tfae:

1. *the p_n admit symmetric circuits of poly orbit-size,*
2. *every p_n is a linear combination of $\hom$-polynomials of graphs of bounded treewidth.*

Computational complexity meets graph structure

Theorem (Curticapean, Dell, & Marx (2017))

Unless $\#\mathrm{W}[1] = \mathrm{FPT}$*, for a collection* $\mathcal{P}$ *of graph motif parameters* $p(\star) = \sum \alpha_F \hom(F, \star)$*, tfae:*

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    - $\hom$-expansion of a graph parameter is not unique
    - no complexity monotonicity

$(\mathrm{perm}_n)_{n \in \mathbb{N}} \notin$

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Definition (Dawar & Wang (2017))

A sequence of symmetric polynomials p_n has *counting width* $\leq k$ if, for all n-vertex graphs G and H,

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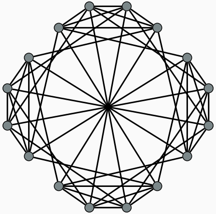
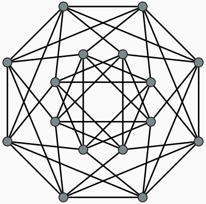
CFI graphs allow to prove counting width lower bounds.



$\equiv_k$



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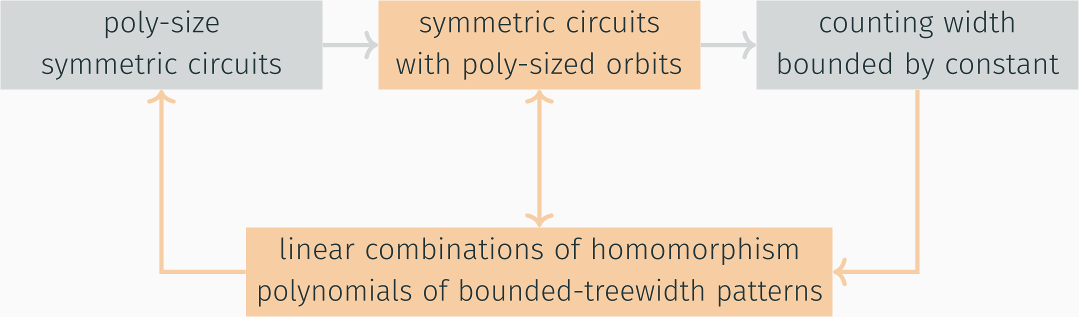

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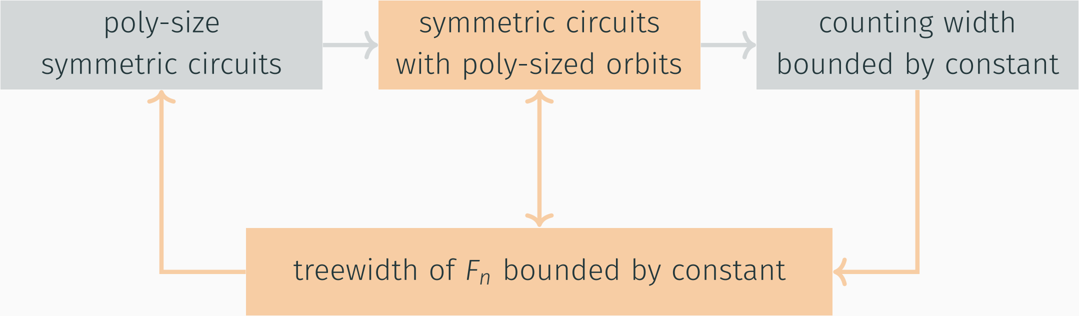

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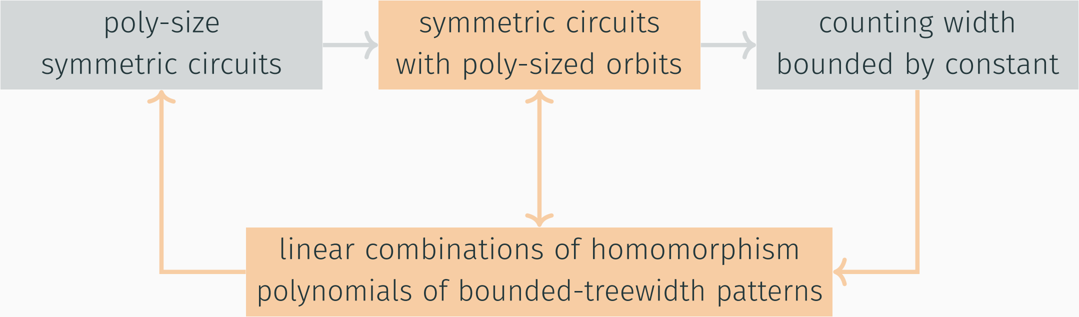

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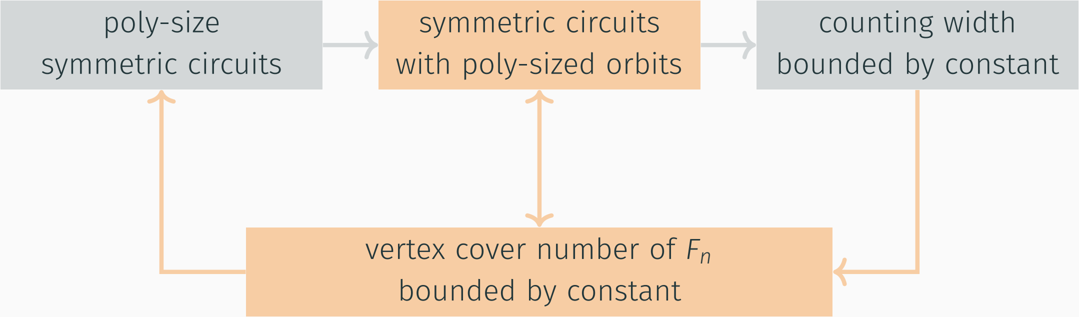

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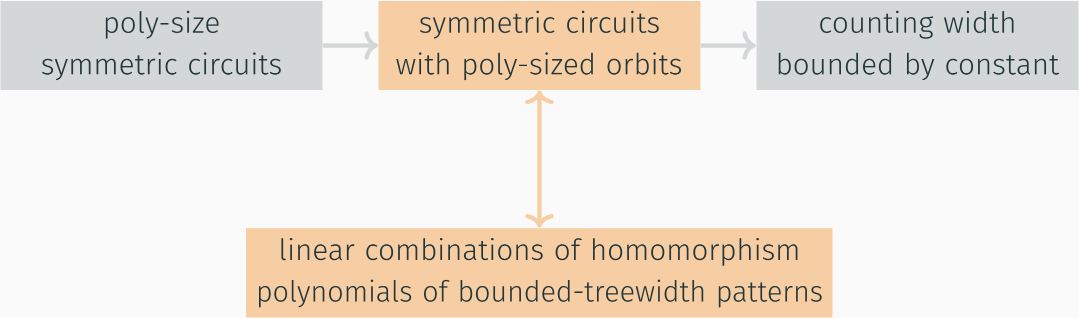

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