NPA Hierarchy for Quantum Isomorphism and Homomorphism Indistinguishability

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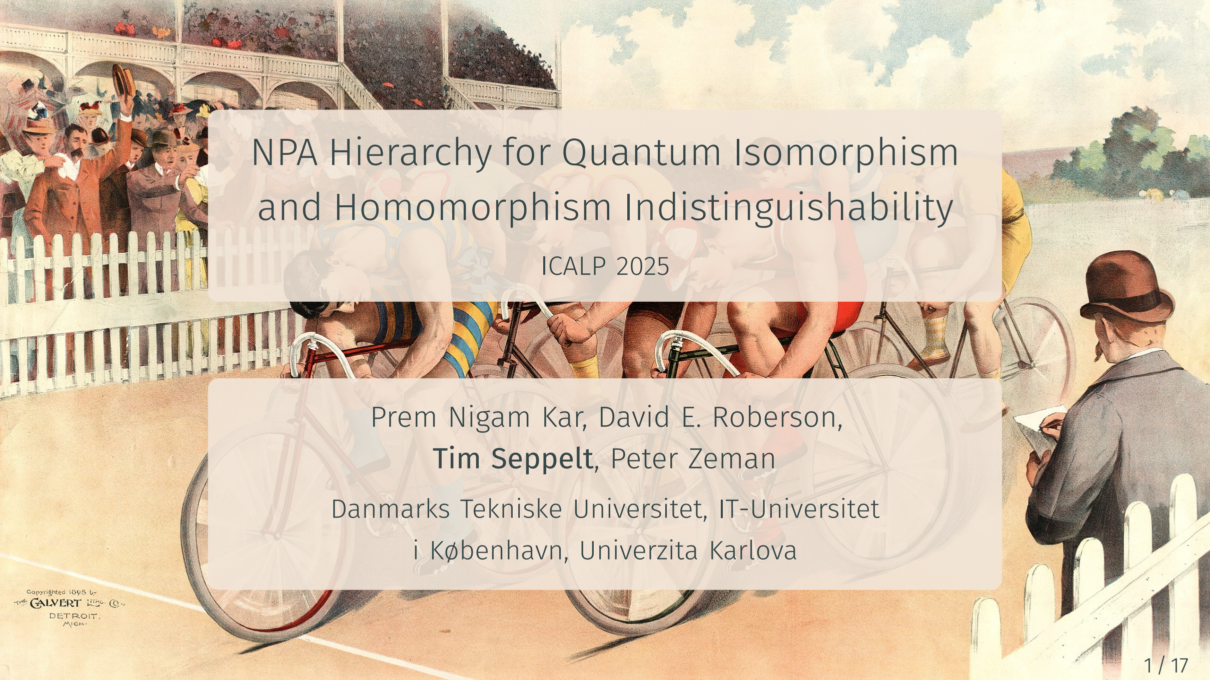

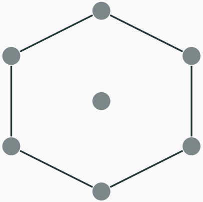

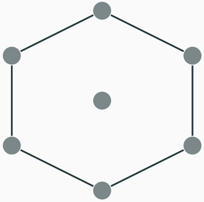

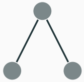

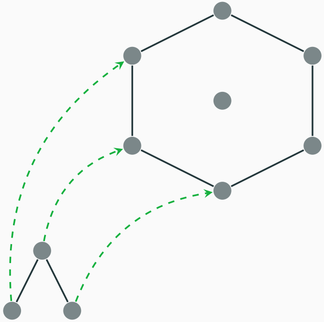

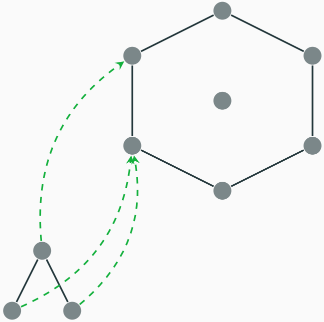

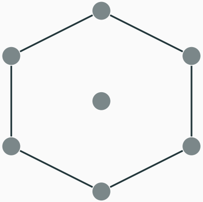

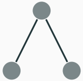

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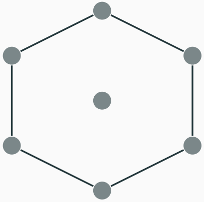

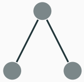

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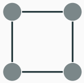

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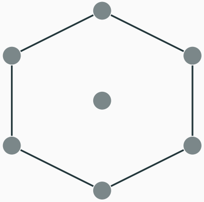
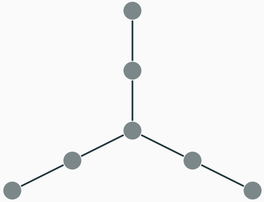
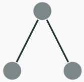
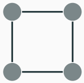
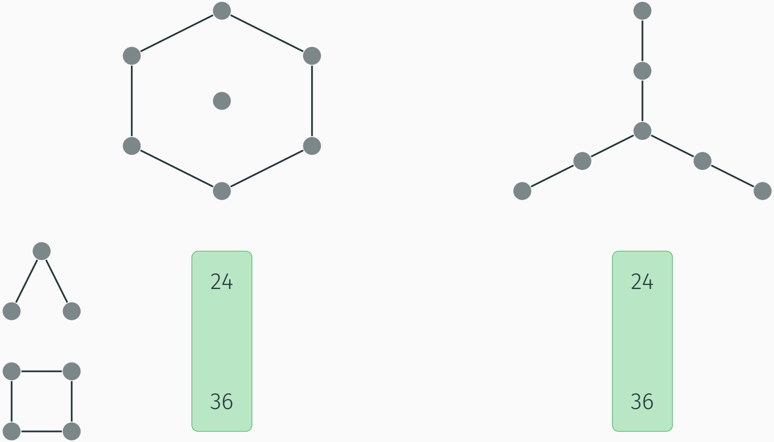
		
	24	24
	36	36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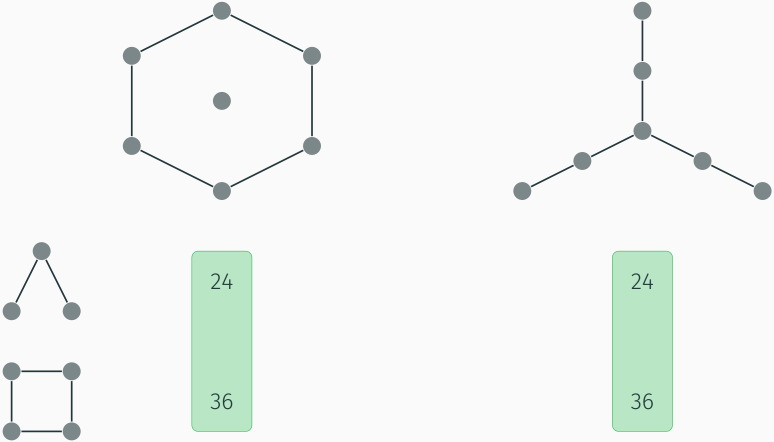
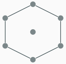
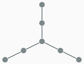


The graphs  and  are **homomorphism indistinguishable** over {, }.

graph class $\mathcal{F}$	**relation $\equiv_{\mathcal{F}}$**	
all graphs	isomorphism	Lovász (1967)

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	Graph Neural Networks	Xu, Hu, Leskovec, & Jegelka (2018); Morris, Ritzert, Fey, Hamilton, Lenssen, Rattan, & Grohe (2019)

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- The emerging theory of homomorphism indistinguishability describes the distinguishing power and complexity of these relations in terms of properties of the corresponding graph classes.

Homomorphism Indistinguishability Zoo

`tseppelt.github.io/homind-database`

Graph classes and their homomorphism indistinguishability properties.

Theorem (Mančinska & Roberson (2020))

Two graphs are quantum isomorphic if, and only if, they are homomorphism indistinguishable over all planar graphs.

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- Proof relies on Woronowicz's Tannaka–Krein duality for compact matrix quantum groups.

  'Needless to say, the quantum groups [...] don't exist as concrete objects.'—Banica, Bichon, & Collins (2007)

Two-player game in which Alice and Bob seek to convince a referee that $G \cong H$.

Referee

Alice

Bob

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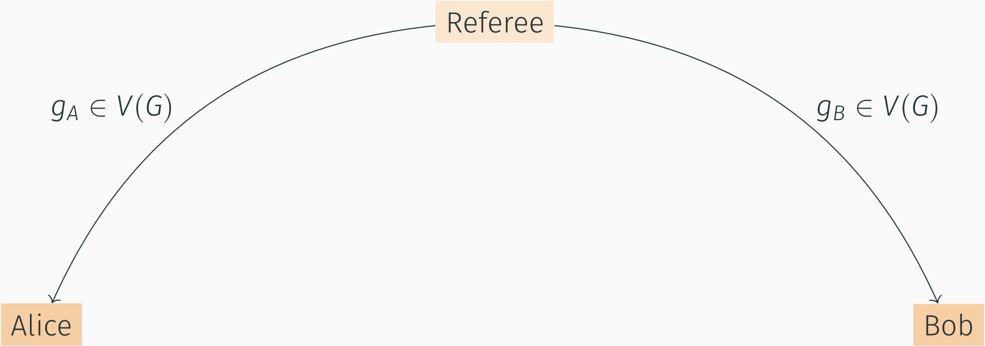

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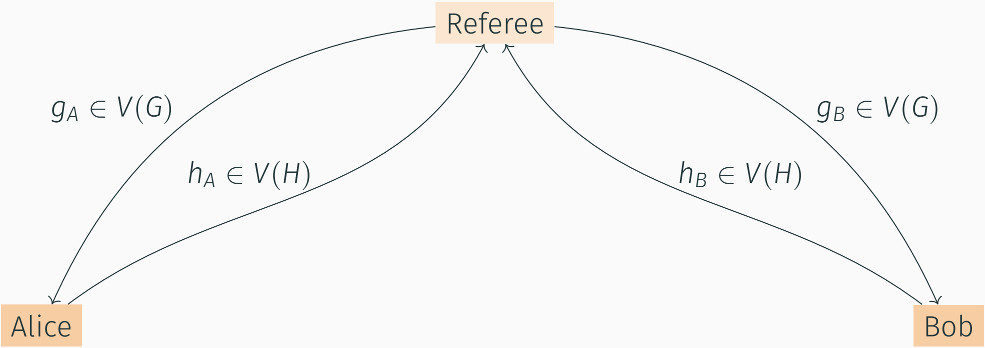

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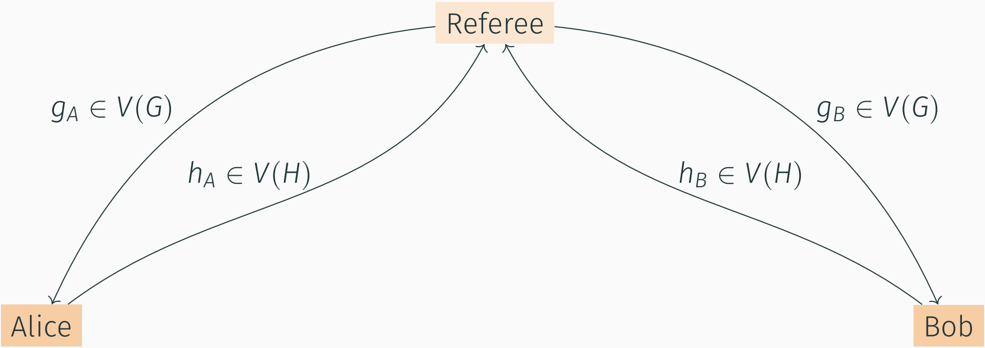

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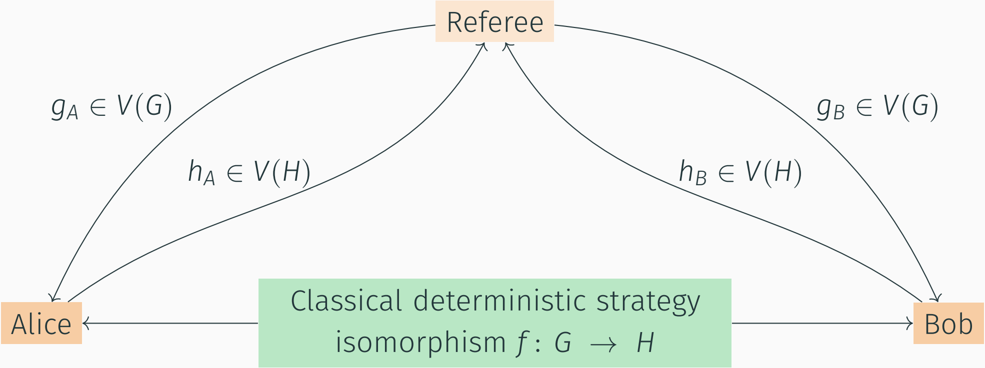

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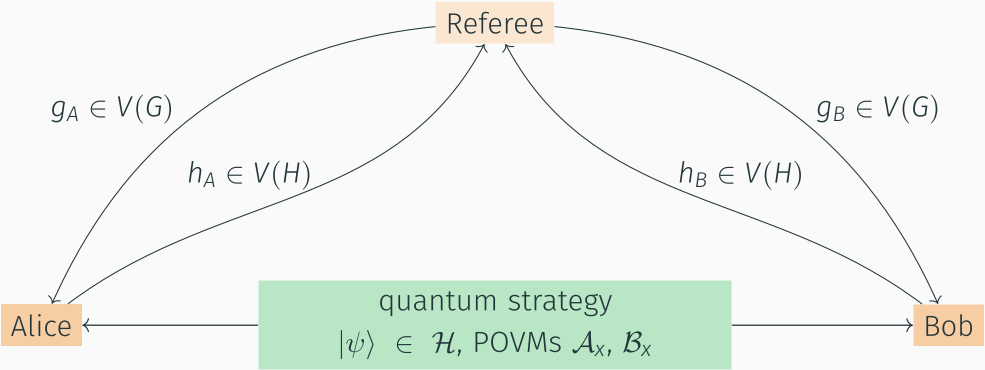

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Extend arguments due to Roberson & Seppelt (2023); Grohe, Rattan, & Seppelt (2022); Mančinska, Roberson, & Varvitsiotis (2023).

Equations

homomorphism tensors,
algebraic operations

Graph Class

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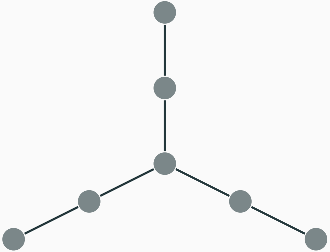
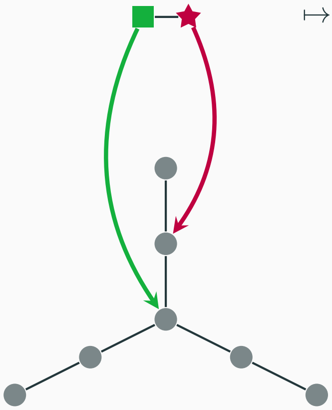
$$\mapsto \left\{ \begin{matrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{matrix} \right.$$

$$\mapsto \begin{matrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}$$





$$\mapsto \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

series composition

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 $\cdot$  $=$ 

$$
\left\{\begin{matrix}
0 & 1 & 0 & 1 & 0 & 1 & 0\\
1 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 1\\
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\end{matrix}\right.
\qquad
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0 & 0 & 0 & 0 & 0 & 1 & 0
\end{matrix}\right.
\qquad
\left\{\begin{matrix}
3 & 0 & 1 & 0 & 1 & 0 & 1\\
0 & 2 & 0 & 1 & 0 & 1 & 0\\
1 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 2 & 0 & 1 & 0\\
1 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 1 & 0 & 1 & 0 & 2 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 1
\end{matrix}\right.
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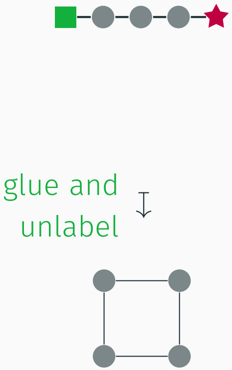
$$\downarrow \qquad \downarrow \qquad \downarrow$$

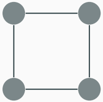
$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

matrix product


glue and
unlabel

$$\mapsto \begin{pmatrix} 12 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 6 & 0 & 5 & 0 & 5 & 0 \\ 4 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 5 & 0 & 6 & 0 & 5 & 0 \\ 4 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 5 & 0 & 5 & 0 & 6 & 0 \\ 4 & 0 & 1 & 0 & 1 & 0 & 2 \end{pmatrix}$$

glue and unlabel $\downarrow$

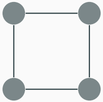


$\mapsto$

12	0	4	0	4	0	4
0	6	0	5	0	5	0
4	0	2	0	1	0	1
0	5	0	6	0	5	0
4	0	1	0	2	0	1
0	5	0	5	0	6	0
4	0	1	0	1	0	2

glue and unlabel $\downarrow$

$\downarrow$ trace



$\mapsto$ 36

Theorem

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Theorem (Specht (1940))

For symmetric matrices A and B, there is an orthogonal matrix X such that $XA = BX$ if, and only if, $\mathrm{tr}(A^n) = \mathrm{tr}(B^n)$ for all $n \in \mathbb{N}$.

Equations

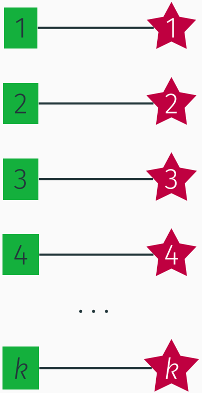
homomorphism tensors,
algebraic operations

→

Graph Class

bilabelled graphs,
combinatorial operations

Let $\mathcal{M}_k$ denote the minors of the bilabelled k-matching.



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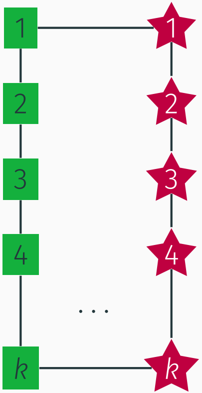
Let $\mathcal{C}_k$ denote the minors of the bilabelled $2k$-cycle.



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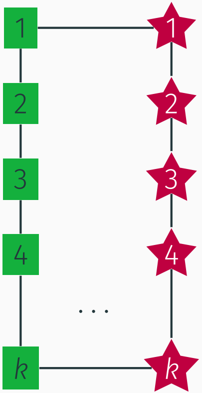
The class $\mathcal{P}_k$ is generated by $\mathcal{C}_k \cup \mathcal{M}_k$ under

- series composition,
- parallel composition with graphs in $\mathcal{C}_k$,
- cyclic permutations of labels,
- transposition.



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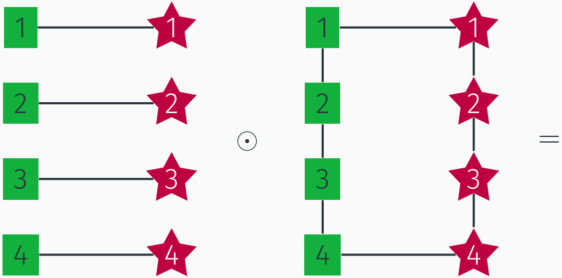

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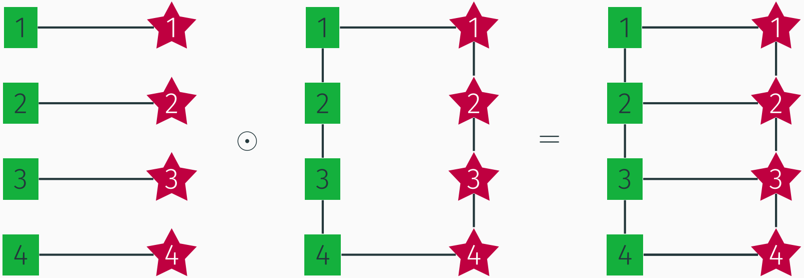

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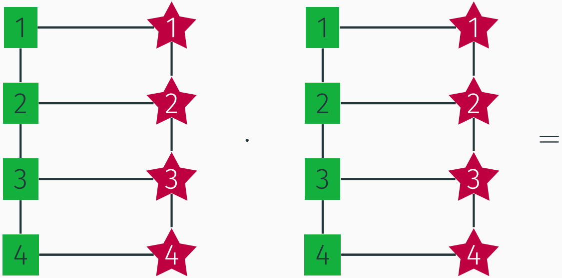

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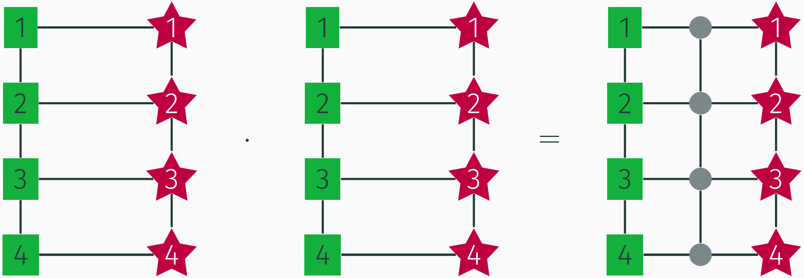

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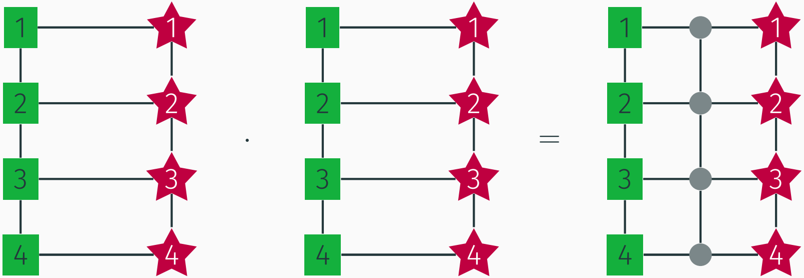

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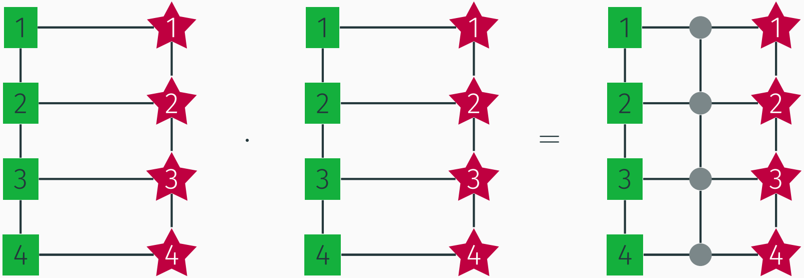

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$\mathcal{P}_k$ has treewidth $\leq 3k - 1$.

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For $k \geq 1$, the level-k NPA relaxation of quantum isomorphism is feasible for two graphs if, and only if, they are homomorphism indistinguishable over $\mathcal{P}_k$.

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Corollary (Distinguishing Power of NPA for quantum isomorphism)

There are non-quantum-isomorphic graphs of size $72k^2$ for which the level-k NPA relaxation is feasible.

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