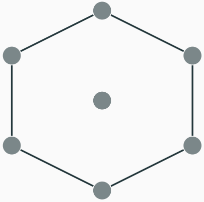
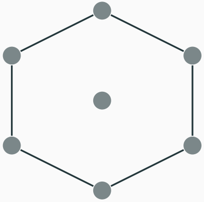
Homomorphism Indistinguishability

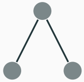
Simons Institute, 12 August 2025

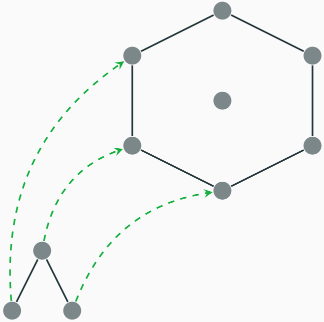
Tim Seppelt

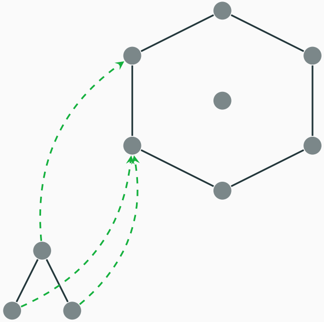
IT-UNIVERSITETET I KØBENHAVN

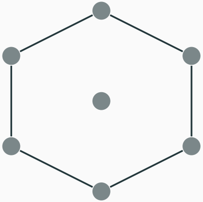


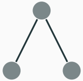












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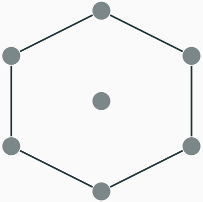

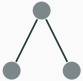

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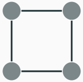

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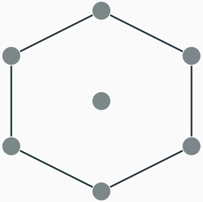
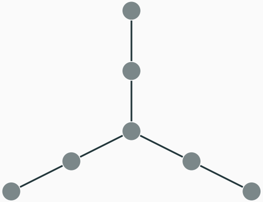
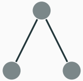
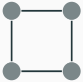
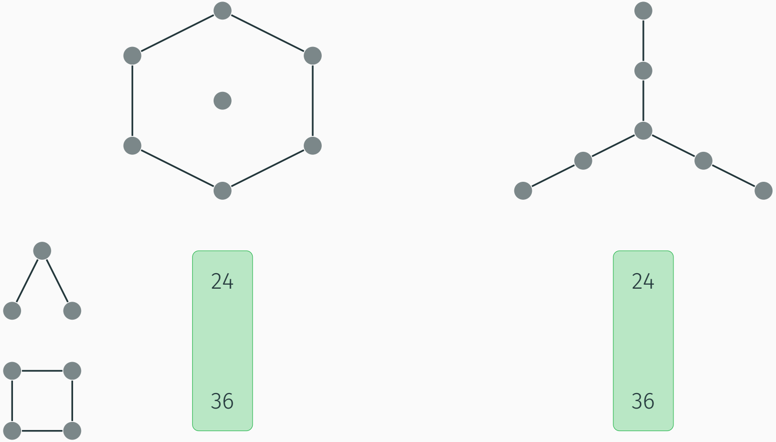
		
	24	24
	36	36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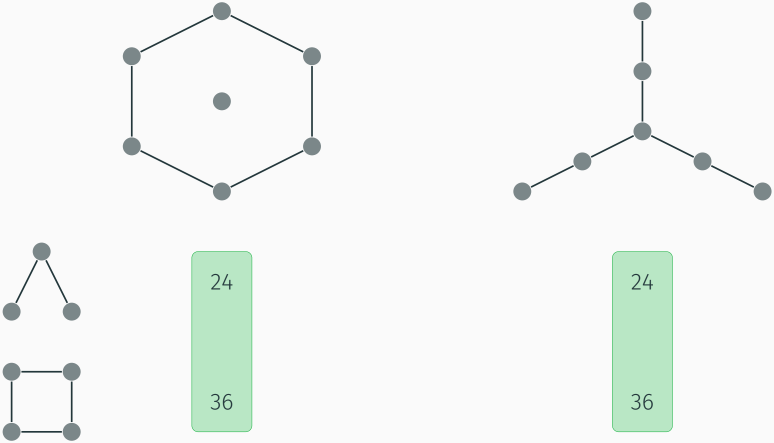
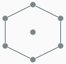
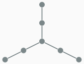


The graphs  and  are **homomorphism indistinguishable** over $\{$$\}$.

graph class $\mathcal{F}$	**relation $\equiv_{\mathcal{F}}$**	
all graphs	isomorphism	Lovász (1967)

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	$\mathbb{P}_k$-coKleisli isomorphism	Dawar, Jakl, & Reggio (2021)

Theorem (Xu, Hu, Leskovec, & Jegelka (2018); Dvořák (2010))

Two graphs are not distinguished by any MPNN if, and only if, they are homomorphism indistinguishable over all trees.

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- Generalisations to MPNN variants due to Morris, Ritzert, Fey, Hamilton, Lenssen, Rattan, & Grohe (2019); Zhang, Gai, Du, Ye, He, & Wang (2024); Gai, Du, Zhang, Maron, & Wang (2025).

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- Insights into distinguishing power of such architectures.

Weisfeiler–Leman

↕

homomorphism indistinguishability

↕

$XA_G = A_H X$

Characterisations

How to characterise $\equiv_{\mathcal{F}}$?

Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?

Complexity

How to test $\equiv_{\mathcal{F}}$?

Characterisations

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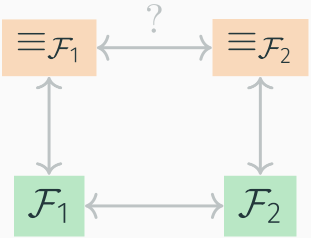

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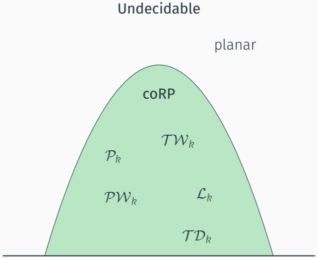

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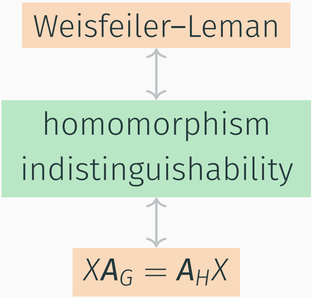

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Observation

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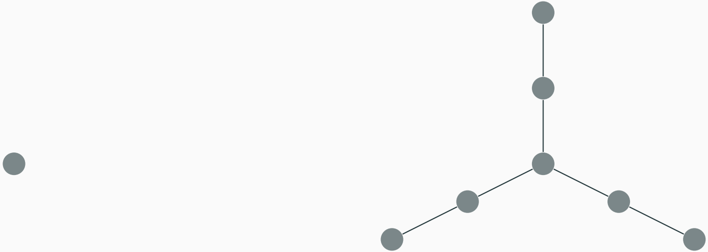

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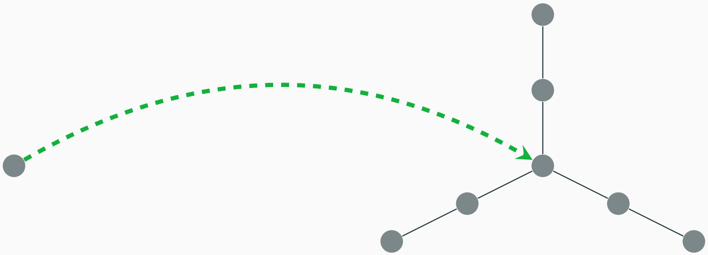

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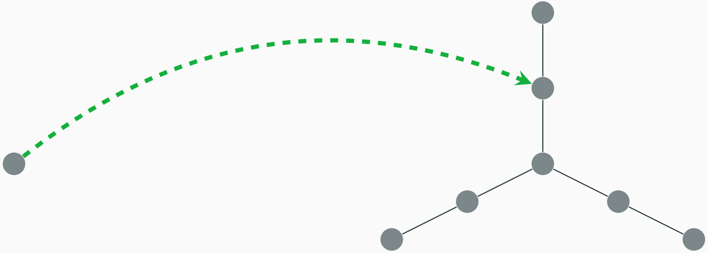

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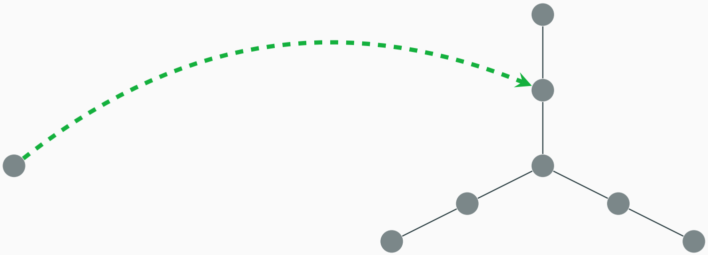
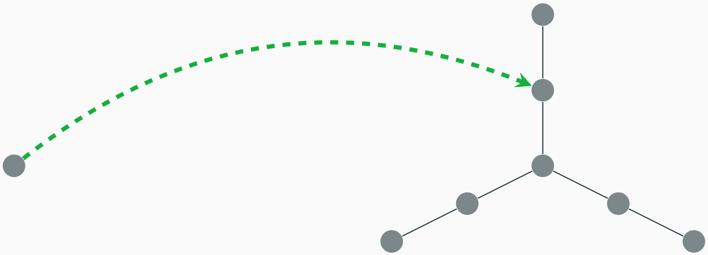


$$\hom(●, G) = |V(G)|.$$

Observation

Two graphs have the same number of vertices if, and only if, they are homomorphism indistinguishable over {●}.



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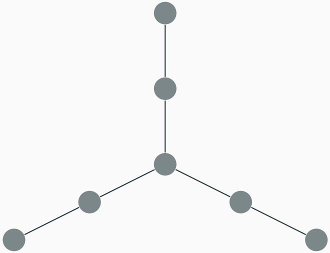

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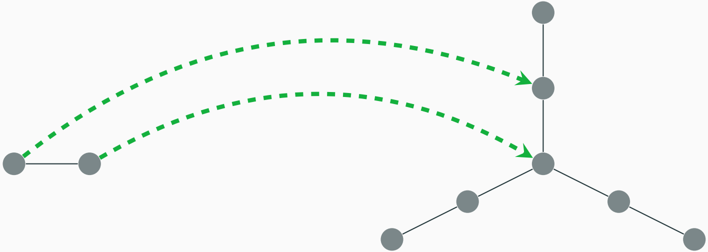

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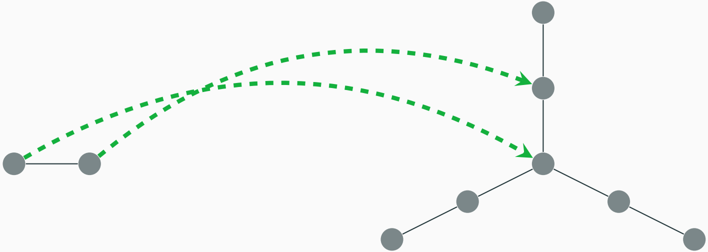

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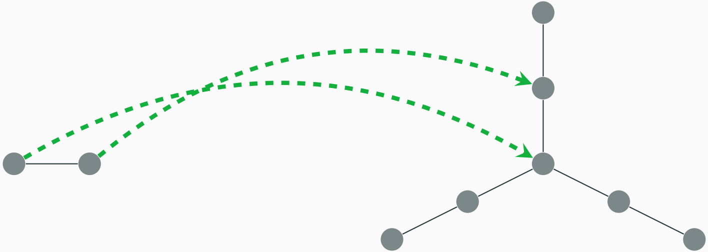
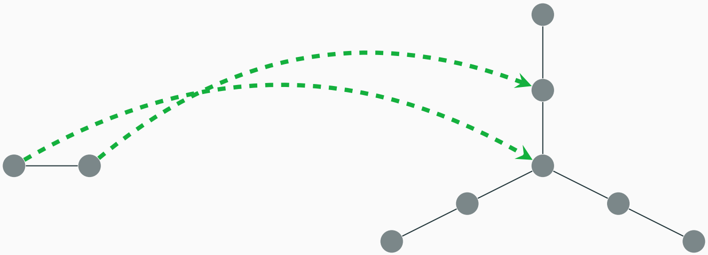


$$\hom(\text{●–●}, G) = 2|E(G)|.$$

Observation

Two graphs have the same number of edges if, and only if, they are homomorphism indistinguishable over {●–●}.



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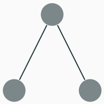

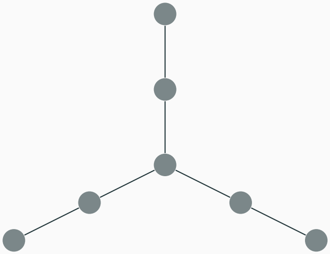

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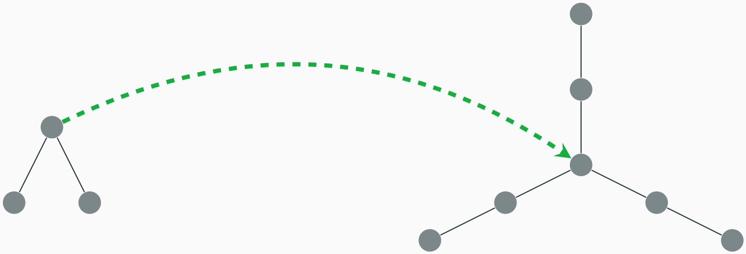

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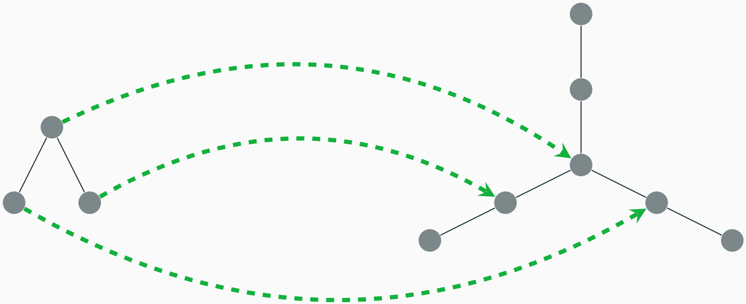

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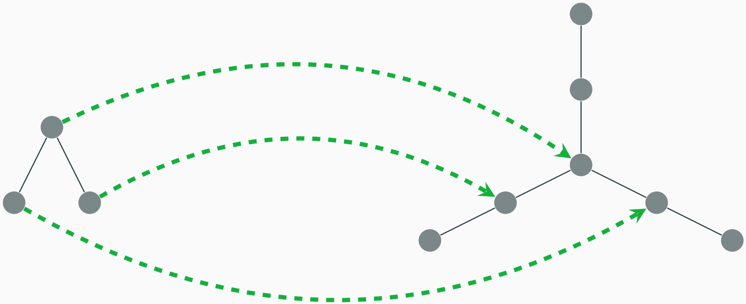
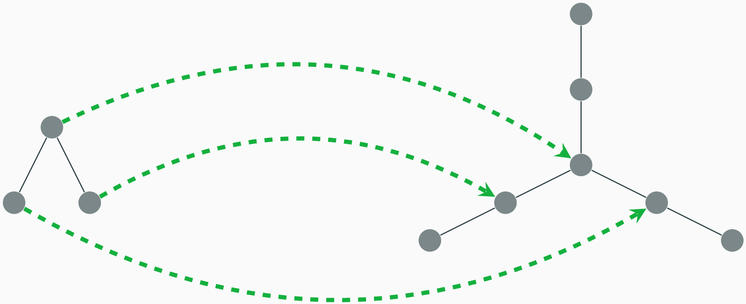


$$\hom(S_\ell, G) = \sum_{v \in V(G)} (\deg(v))^\ell$$

Observation

Two graphs have the same degree sequence if, and only if, they are homomorphism indistinguishable over stars.



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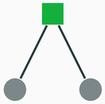
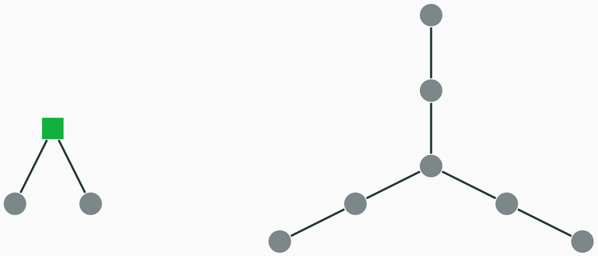
Theorem (Dvořák (2010); Dell, Grohe, & Rattan (2018))

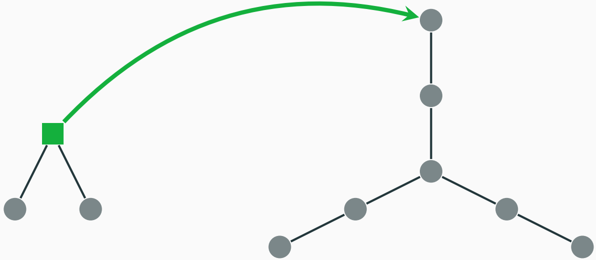
Two graphs are not distinguished by Colour Refinement if, and only if, they are homomorphism indistinguishable over trees.

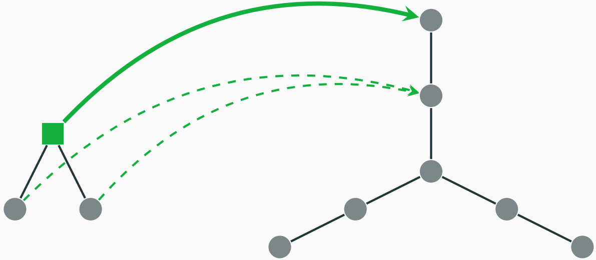
Theorem (Mančinska & Roberson (2020))

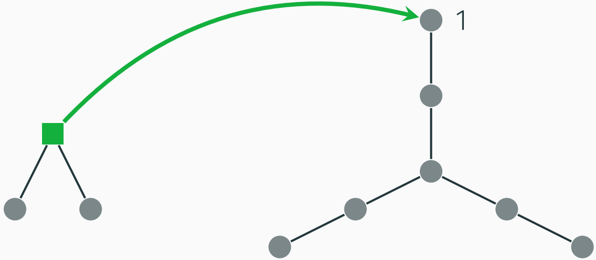
Two graphs are quantum isomorphic if, and only if, they are homomorphism indistinguishable over all planar graphs.







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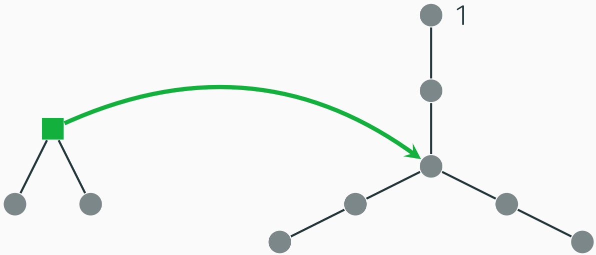
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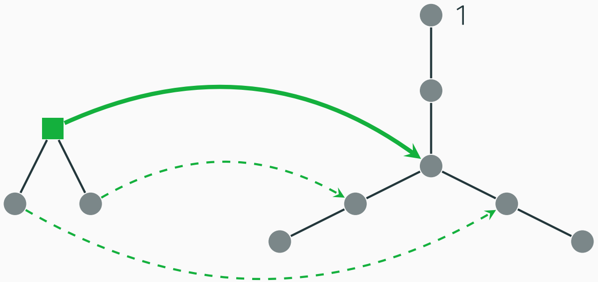
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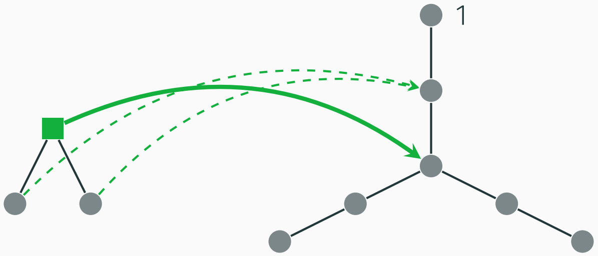
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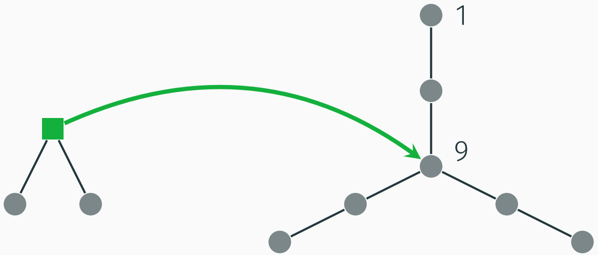
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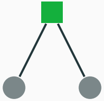
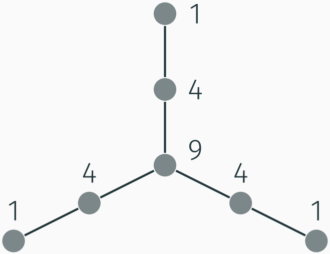
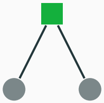
labelled graphs $\rightarrow$ homomorphism vectors $\subseteq \mathbb{R}^{V(G)}$



$\mapsto$





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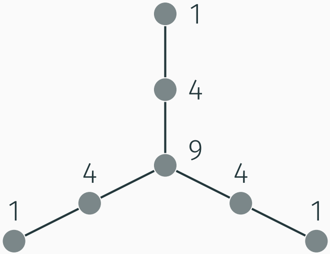

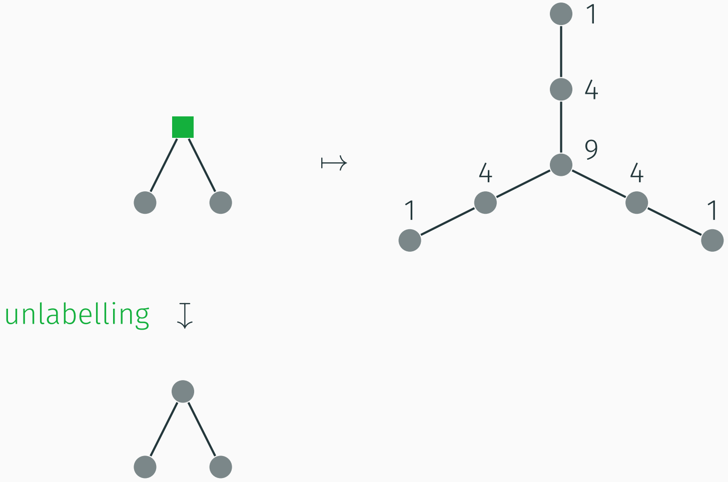
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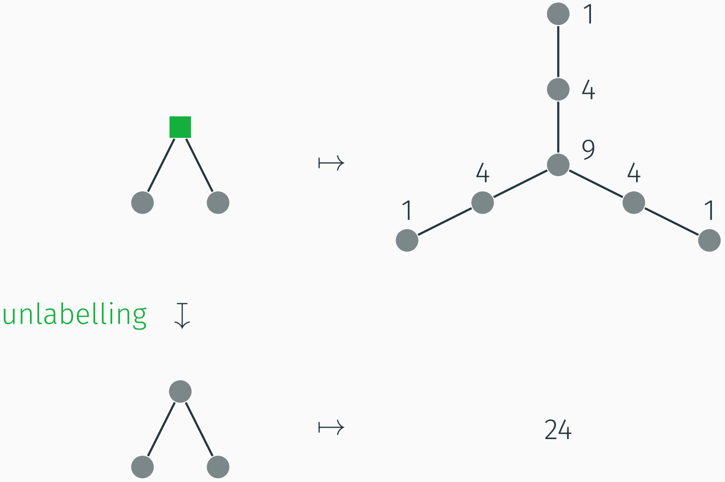
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unlabelling
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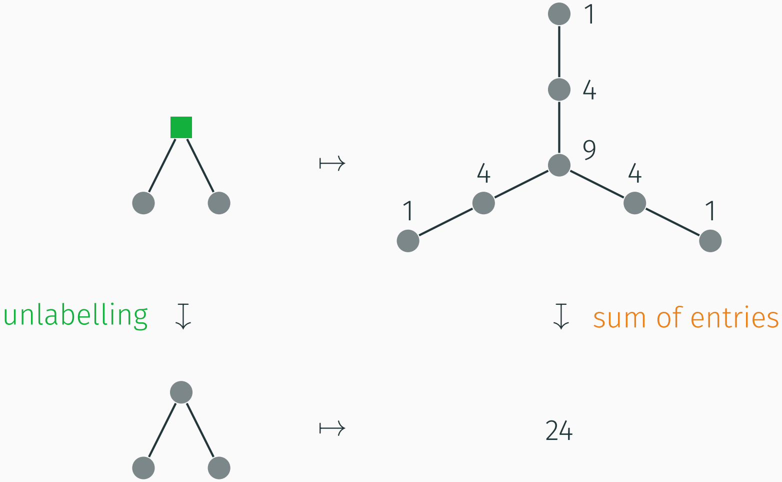
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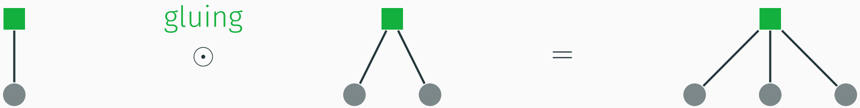
sum of entries
24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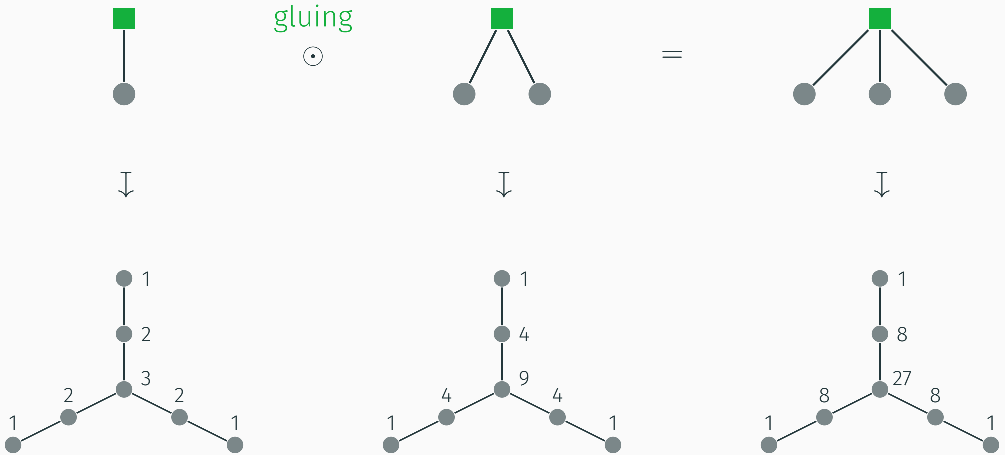
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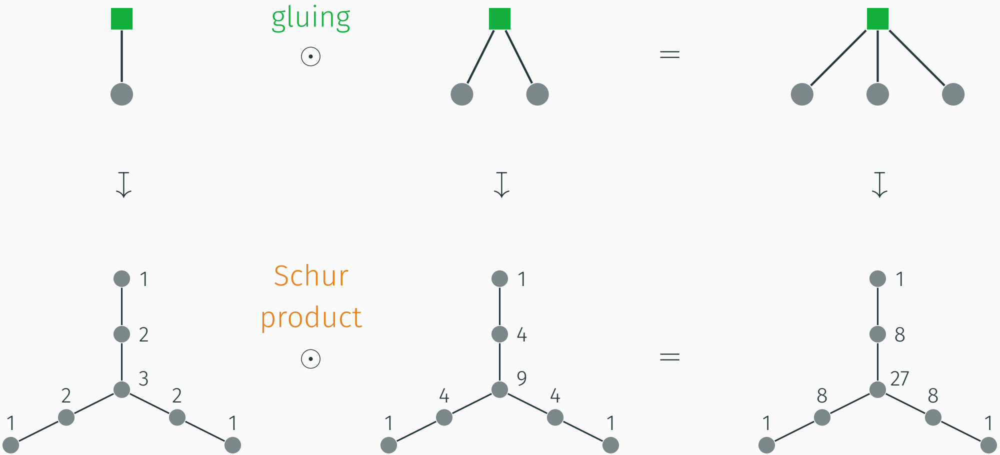
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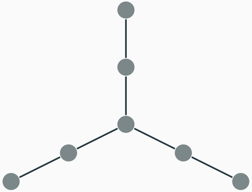

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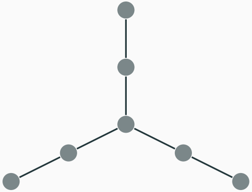

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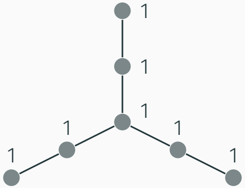

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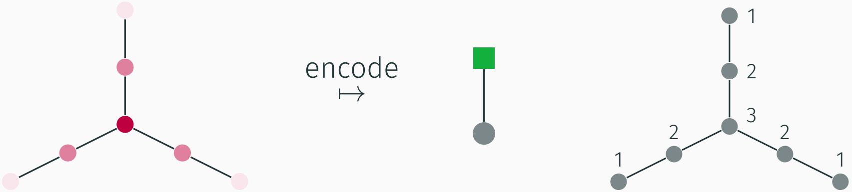

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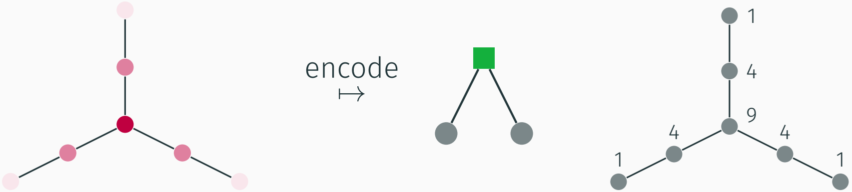

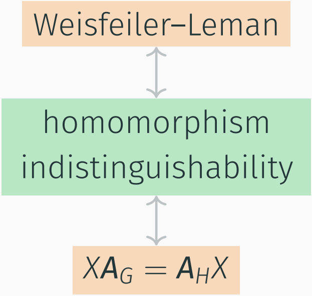

Characterisations

How to characterise $\equiv_{\mathcal{F}}$?

Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?

Complexity

How to test $\equiv_{\mathcal{F}}$?

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A matrix $X = (x_{ij})$ over some C^*-algebra is a *quantum permutation matrix* if

$$x_{ij}^2 = x_{ij} = x_{ij}^*, \qquad \sum_k x_{ik} = 1 = \sum_k x_{kj}.$$

$$XA_G = A_H X$$

planar graphs

quantum permutation

Lupini, Mančinska, & Roberson (2020); Mančinska & Roberson (2020); Lovász (1967); Dell, Grohe, & Rattan (2018); Grohe, Rattan, & S. (2022); S. (2024).

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all graphs
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planar graphs
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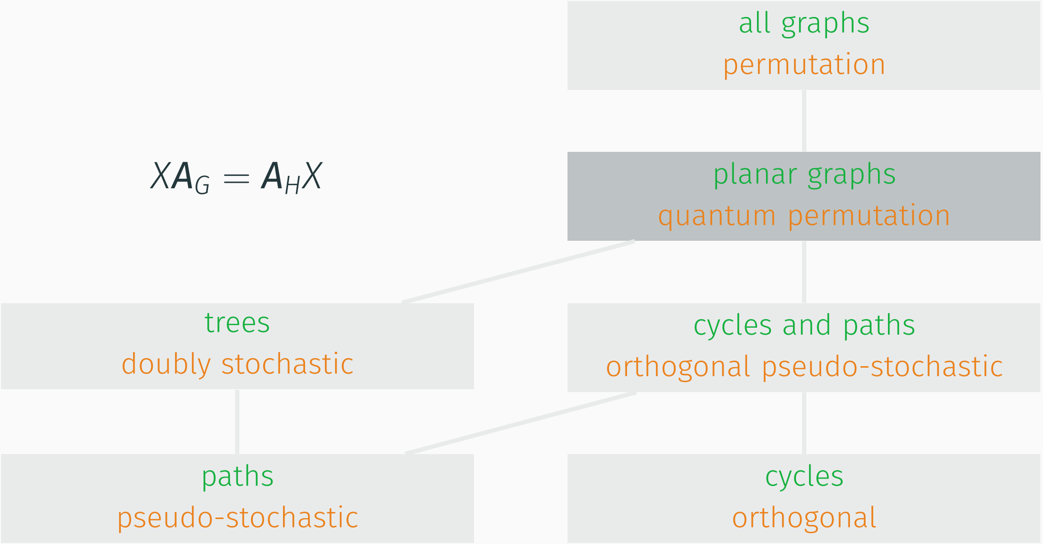

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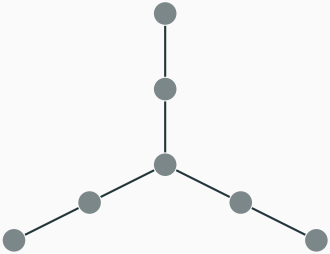
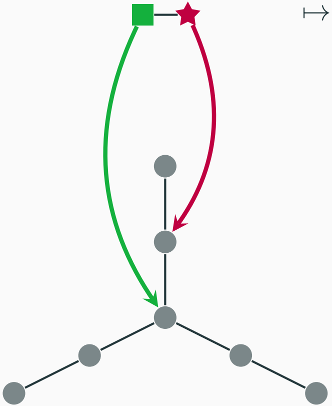
$$\mapsto \begin{matrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}$$

$\mapsto$

$$\begin{matrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}$$





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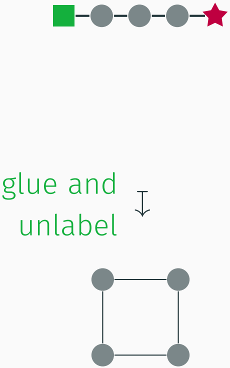
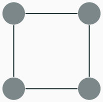

glue and
unlabel

$\mapsto$

12	0	4	0	4	0	4
0	6	0	5	0	5	0
4	0	2	0	1	0	1
0	5	0	6	0	5	0
4	0	1	0	2	0	1
0	5	0	5	0	6	0
4	0	1	0	1	0	2

glue and unlabel $\downarrow$

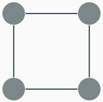


↦

12	0	4	0	4	0	4
0	6	0	5	0	5	0
4	0	2	0	1	0	1
0	5	0	6	0	5	0
4	0	1	0	2	0	1
0	5	0	5	0	6	0
4	0	1	0	1	0	2

glue and unlabel ↧

↧ trace



↦ 36

Theorem

There is an orthogonal matrix X such that $X\boldsymbol{A}_G = \boldsymbol{A}_H X$ if, and only if, G and H are homomorphism indistinguishable over all cycles.

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Theorem (Specht (1940))

For symmetric matrices A and B, there is an orthogonal matrix X such that $XA = BX$ if, and only if, $\mathrm{tr}(A^n) = \mathrm{tr}(B^n)$ for all $n \in \mathbb{N}$.

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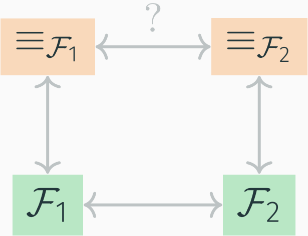
Check out Kar, Roberson, S., & Zeman (2025) for an elementary proof!

Characterisations

How to characterise $\equiv_{\mathcal{F}}$?



Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?

Complexity

How to test $\equiv_{\mathcal{F}}$?

Are all cospectral graphs isomorphic?

Günthard & Primas (1956)

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Homomorphism indistinguishability over cycles is not isomorphism.

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Homomorphism indistinguishability over trees is not isomorphism.

Can $\equiv_{\mathcal{F}}$ be isomorphism for a proper graph class $\mathcal{F}$?

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A graph class $\mathcal{F}$ is homomorphism distinguishing closed if, for all $F' \notin \mathcal{F}$,

$$\text{there exist } G \text{ and } H \text{ such that } G \equiv_{\mathcal{F}} H \text{ and } \hom(F', G) \neq \hom(F', H).$$

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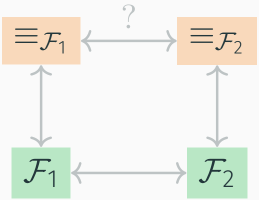
Which graph classes are homomorphism distinguishing closed?

Observation

If $\mathcal{F}_1$ is homomorphism distinguishing closed, then

$$\equiv_{\mathcal{F}_1} \text{ refines } \equiv_{\mathcal{F}_2} \quad \iff \quad \mathcal{F}_1 \text{ is a superclass of } \mathcal{F}_2.$$



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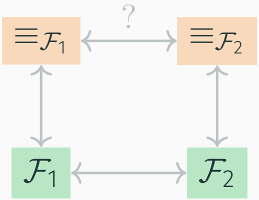

- optimisation — Roberson & S. (2023)
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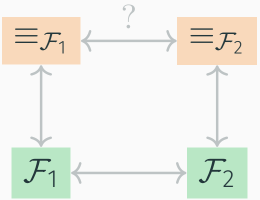

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Definition (Zhang, Gai, Du, Ye, He, & Wang (2024))

The homomorphism distinguishing closed graph class $\mathcal{F}$ is the homomorphism expressivity of $\equiv_{\mathcal{F}}$.

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Conjecture (Roberson (2022))

Every minor-closed union-closed graph class is homomorphism distinguishing closed.

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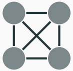
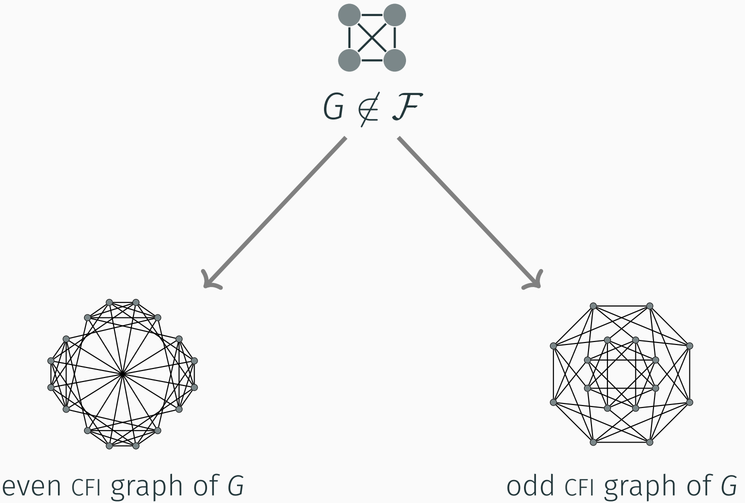
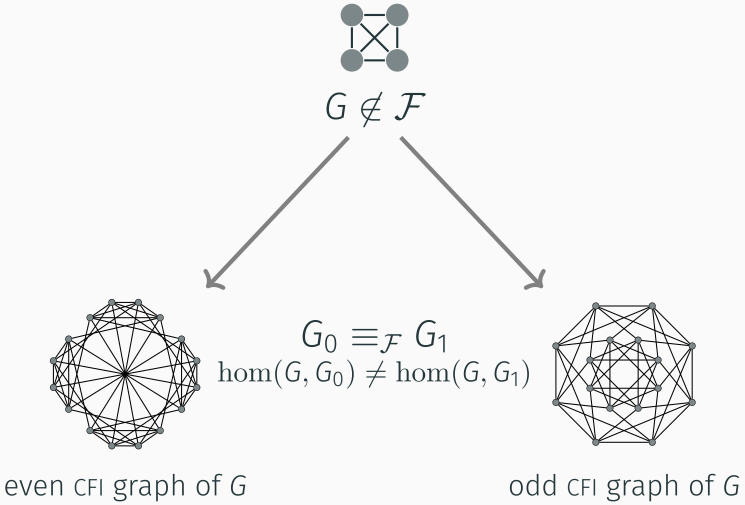
- planar graphs — Roberson (2022)
- treewidth $\leq k$ — Neuen (2024)
- treedepth $\leq q$ — Fluck, S., & Spitzer (2024)
- k-pebble forest cover of depth $\leq q$ — Adler & Fluck (2024)
- pathwidth $\leq k$ — S. (2024)
- essentially finite graph classes — S. (2023)
- outerplanar graphs — Neuen & S. (2024)



$$G \notin \mathcal{F}$$


$G \notin \mathcal{F}$
even CFI graph of G
odd CFI graph of G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$G_0 \equiv_{\mathcal{F}} G_1$
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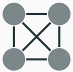

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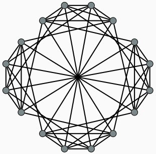
Graph searching game

Robber



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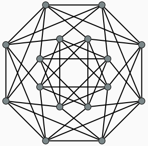
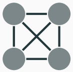


even CFI graph of G

odd CFI graph of G



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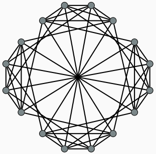

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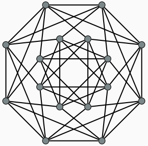

even CFI graph of G

odd CFI graph of G

Graph searching game

Robber

Model comparison game

Duplicator

Theorem (Roberson (2022))

For a connected graph G and any graph F, the following are equivalent:

1. $\hom(F, G_0) \neq \hom(F, G_1)$,
2. *there exists a weak oddomorphism* $F \to G$.

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For a connected graph G and any graph F, the following are equivalent:

1. $\hom(F, G_0) \neq \hom(F, G_1)$,
2. *there exists a weak oddomorphism* $F \to G$.

If $F \to G$ is a weak oddomorphism, then

- $\mathrm{tw}(F) \geq \mathrm{tw}(G)$, Neuen (2024)
- $\mathrm{td}(F) \geq \mathrm{td}(G)$, Fluck, S., & Spitzer (2024)
- $\mathrm{pw}(F) \geq \mathrm{pw}(G)$, S. (2024)
- F planar $\implies$ G planar, Roberson (2022)
- $\Delta(F) \geq \Delta(G)$, Roberson (2022)
- F outerplanar $\implies$ G outerplanar. Neuen & S. (2024)

Theorem (S. (2023))

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- Typical graph isomorphism relaxations are preserved under complements.
- Towards a **theory of homomorphism indistinguishability**, we can focus on minor-closed graph classes.

Theorem (Roberson (2022))

There are uncountably many homomorphism distinguishing closed graph classes.

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Theorem (van Dobben de Bruyn, Marquès, Roberson, S., Zeman (2025+))

There is a topology whose closed sets are precisely the homomorphism distinguishing closed sets.

Characterisations

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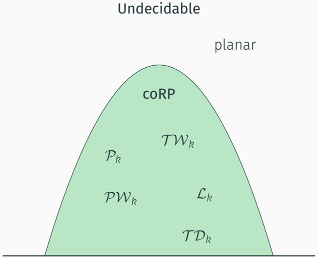
Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?



Complexity

How to test $\equiv_{\mathcal{F}}$?

Let $\mathcal{F}$ be minor-closed and proper.

$\textsc{HomInd}(\mathcal{F})$	
Input	Graphs G and H.
Decide	$G \equiv_{\mathcal{F}} H$.

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HomInd($\mathcal{F}$)

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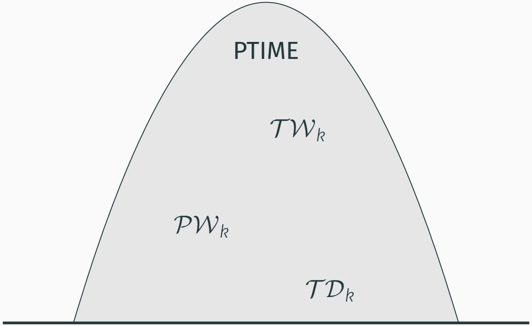

Dell, Grohe, & Rattan (2018); Dvořák (2010); Grohe (2020); Grohe, Rattan, S. (2022)

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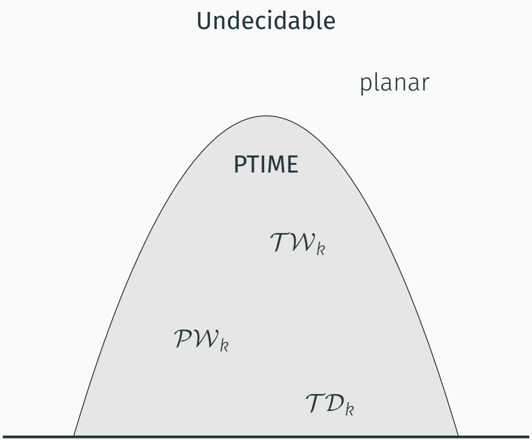

Dell, Grohe, & Rattan (2018); Dvořák (2010); Grohe (2020); Grohe, Rattan, S. (2022); Mančinska & Roberson (2020); Atserias, Mančinska, Roberson, Šámal, Severini, & Varvitsiotis (2019); Slofstra (2019)

Theorem (S. (2024))

If $\mathcal{F}$ is minor-closed and of bounded treewidth, then $\textsc{HomInd}(\mathcal{F})$ *is in* coRP.

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Reduction to equivalence testing for $\mathbb{Q}$-weighted tree automata, which is LOGSPACE interreducible with arithmetic circuit identity testing.

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Theorem (S. (2025+))

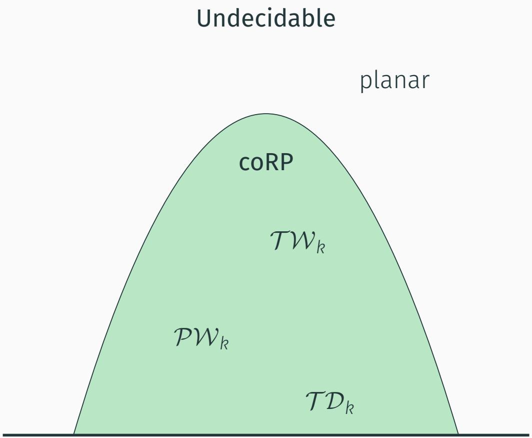
If $\mathcal{F}$ is minor-closed and of bounded pathwidth, then $\textsc{HomInd}(\mathcal{F})$ *is in* NC.

Reduction to equivalence testing for $\mathbb{Q}$-weighted automata. Tzeng (1996)

Let $\mathcal{F}$ be minor-closed and proper.

Theorem (S. (2024))

If $\mathcal{F}$ has bounded treewidth, then HomInd($\mathcal{F}$) *is in* coRP.



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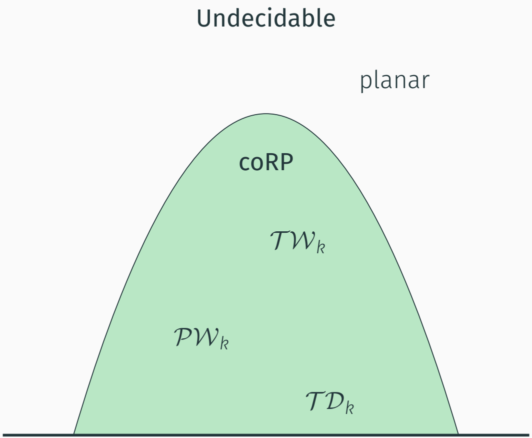

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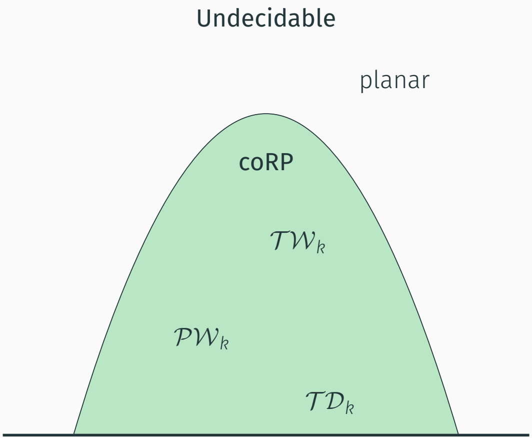
Theorem (S. (2024))

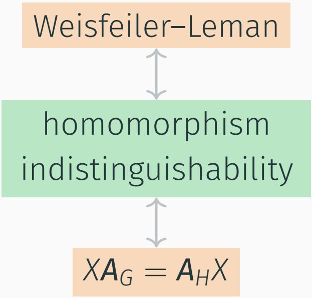
If $\mathcal{F}$ has bounded treewidth, then HomInd($\mathcal{F}$) *is in* coRP.

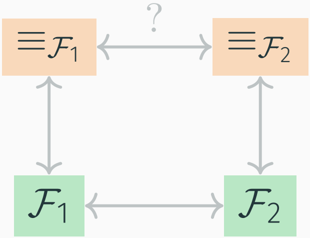
Conjecture (S. (2024))

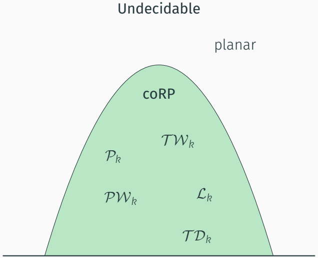
If $\mathcal{F}$ has bounded treewidth, then HomInd($\mathcal{F}$) *is in* PTIME.

Otherwise, HomInd($\mathcal{F}$) *is undecidable.*





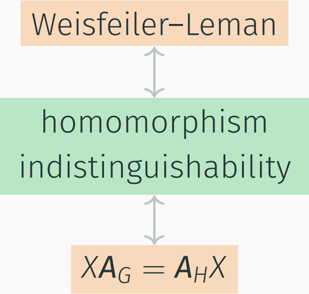




Characterisations
How to characterise $\equiv_{\mathcal{F}}$?

Distinguishing Power
What's the power of $\equiv_{\mathcal{F}}$?

Complexity
How to test $\equiv_{\mathcal{F}}$?



Characterisations

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Weisfeiler–Leman

↕

homomorphism indistinguishability

↕

$XA_G = A_H X$

Characterisations

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- Results by Lovász (1967); Dvořák (2010); Mančinska & Roberson (2020)

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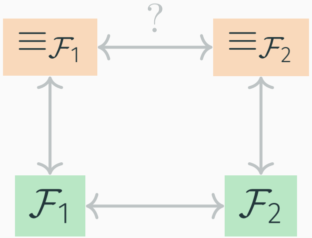
Characterisations

How to characterise $\equiv_{\mathcal{F}}$?

- Results by Lovász (1967); Dvořák (2010); Mančinska & Roberson (2020)
- **Tools:** labelled graphs and homomorphism vectors



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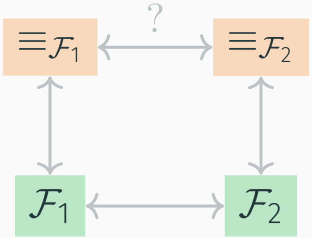

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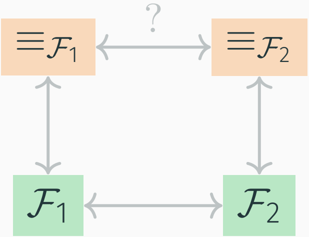
What's the power of $\equiv_{\mathcal{F}}$?

- Comparing graph isomorphism relaxations by comparing graph classes



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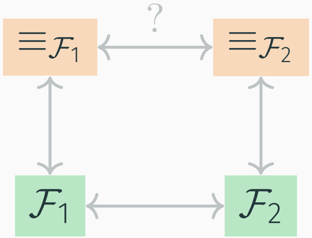
Theory of Homomorphism Indistinguishability



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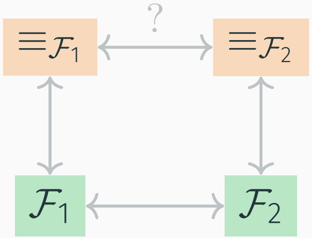
Theory of Homomorphism Indistinguishability

- minor-closed graph classes play a central role.



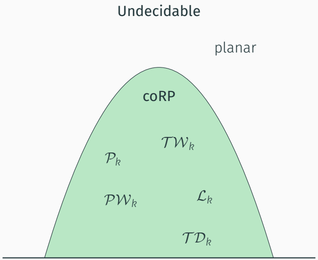
Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?

- Comparing graph isomorphism relaxations by comparing graph classes

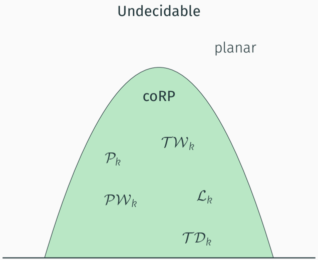
Theory of Homomorphism Indistinguishability

- minor-closed graph classes play a central role.
- **Open:** Roberson's conjecture



Complexity

How to test $\equiv_{\mathcal{F}}$?



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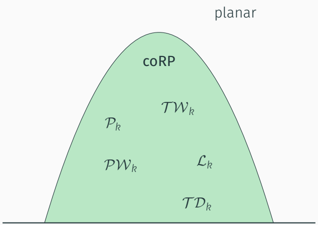
How to test $\equiv_{\mathcal{F}}$?

Theory of Homomorphism Indistinguishability

Undecidable



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How to test $\equiv_{\mathcal{F}}$?

Theory of Homomorphism Indistinguishability

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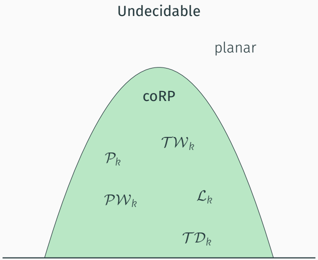

Complexity

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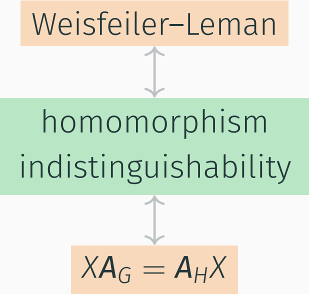
Theory of Homomorphism Indistinguishability

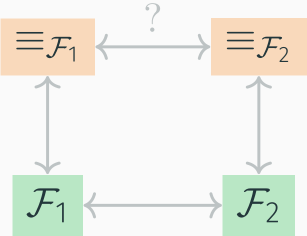
- HomInd($\mathcal{F}$) is in coRP for minor-closed graph classes $\mathcal{F}$ of bounded treewidth.
- **Open:** (un)decidable for proper minor-closed graph classes of unbounded treewidth.

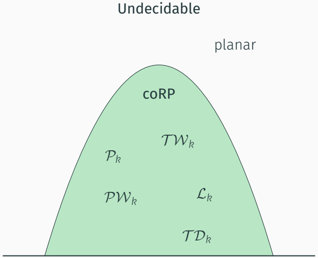
Homomorphism Indistinguishability Zoo

`tseppelt.github.io/homind-database`

Graph classes and their homomorphism indistinguishability properties.







Characterisations

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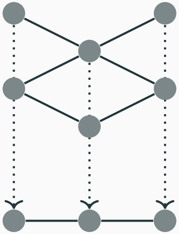
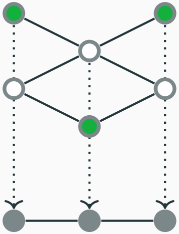
Let $\varphi \colon F \to G$ be a homomorphism.



Let $\varphi\colon F \to G$ be a homomorphism.

A vertex $a \in V(F)$ is φ-even / φ-odd if $|N_F(a) \cap \varphi^{-1}(u)|$ is even / odd for every $u \in N_G(\varphi(a))$.



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φ is an oddomorphism if

- every $a \in V(F)$ is φ-even or φ-odd,
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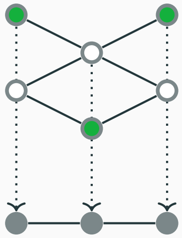

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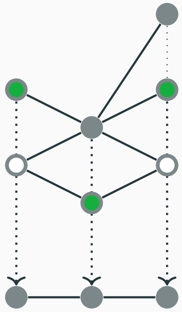

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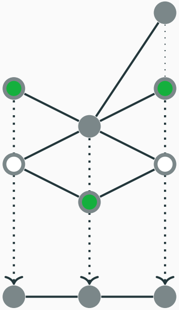

Hypothesis

If $F \to G$ is a weak oddomorphism, then G is a minor of F.

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Title Picture: 'Bicycle race scene. A peloton of six cyclists crosses the finish line in front of a crowded grandstand, observed by a referee.' (1895) by Calvert Lithographic Co., Detroit, Michigan, Public Domain, via Wikimedia Commons. https://commons.wikimedia.org/wiki/File:Bicycle_race_scene,_1895.jpg