# Lower Bounds in Algebraic Complexity via Symmetry and Homomorphism Polynomials

STOC 2026

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Joint work with Prateek Dwivedi (IT-Universitetet i København) and Benedikt Pago (University of Cambridge)

IT-UNIVERSITETET I KØBENHAVN

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Every graph parameter can be written as linear combinations of homomorphism counts…

...and these linear combinations convey complexity-theoretic information.

**Theorem**

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1. Definitions of symmetric algebraic complexity classes.
2. Unconditional symmetric lower bounds.
3. The miraculous world of linear combinations of $\mathrm{hom}$-polynomials.

# Algebraic circuits

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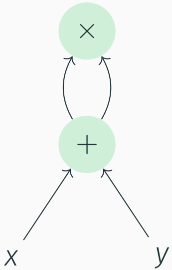


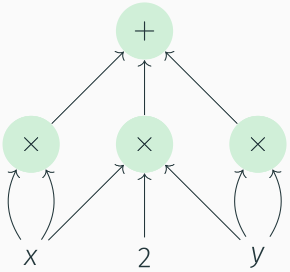

# VP vs. VNP

## Determinant

$$\det_n = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} x_{i,\pi(i)}$$

## Permanent

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**Theorem (Dawar & Wilsenach (2020))**

$\det_n$ *admits poly-size symmetric circuits.*

**Theorem (Dawar & Wilsenach (2020))**

$\operatorname{perm}_n$ *does not admit poly-size symmetric circuits.*

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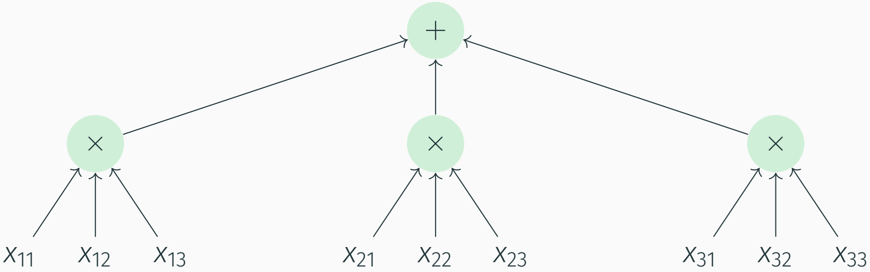

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Symmetric polynomials are functions of $(n, n)$-vertex bipartite graphs.
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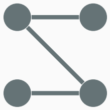

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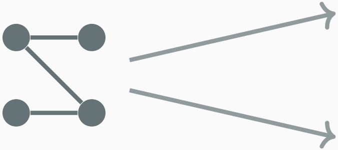

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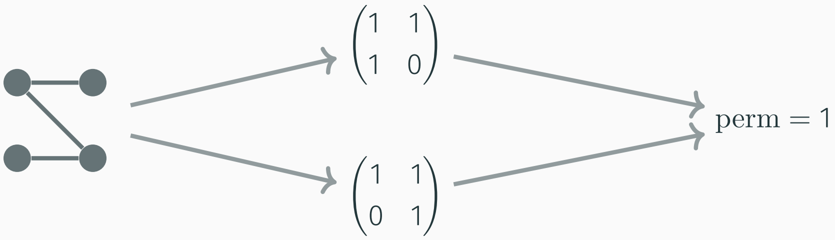

For a bipartite multigraph $F$ and $n \in \mathbb{N}$,

$$\hom_{F,n} := \sum_{h\colon V(F)\to[n]} \prod_{uv\in E(F)} x_{h(u),h(v)}$$

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**Fact**

*A polynomial is symmetric $\iff$ it is a linear combination of $\mathrm{hom}$-polynomials.*

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*A polynomial is symmetric $\iff$ it is a linear combination of $\hom$-polynomials.*

**Corollary**

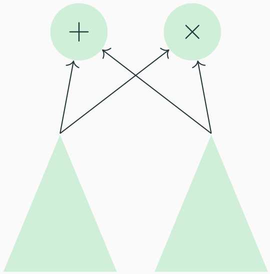
*Every graph parameter can be written as*

$$p_n(\star) = \sum \alpha_{F,n} \hom(F, \star).$$

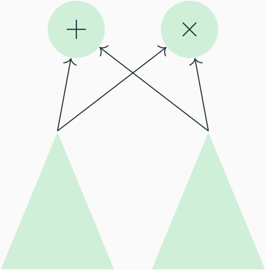
symVP

symmetric circuits of

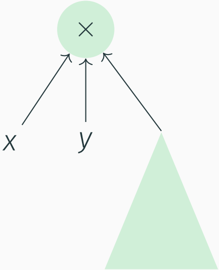
polynomial orbit-size



symVP
symmetric circuits of polynomial orbit-size




symVBP
symmetric skew circuits of polynomial orbit-size



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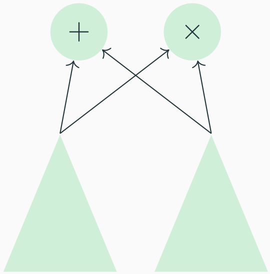

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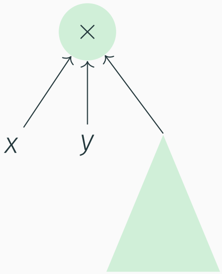
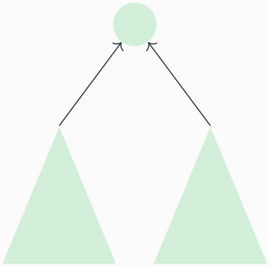



symVF
symmetric formulas of polynomial orbit-size



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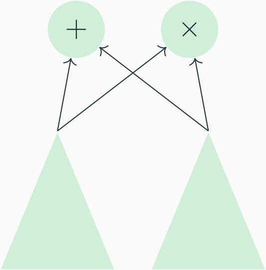

VP

poly-size circuits

symVBP

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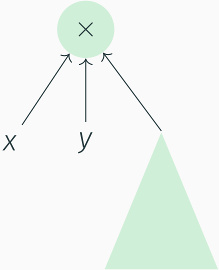


VBP

poly-size skew circuits

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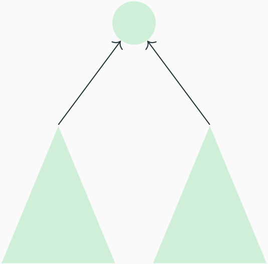

VF

poly-size formulas

| symVP | symVBP | symVF |
| --- | --- | --- |
| symmetric circuits of polynomial orbit-size | symmetric skew circuits of polynomial orbit-size | symmetric formulas of polynomial orbit-size |

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Dawar, Pago, S. (2026)

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linear combinations of $\hom$-polynomials for patterns $F$ with $\mathrm{td}(F) \geq \mathrm{pw}(F) \geq \mathrm{tw}(F) \in \Omega(n)$

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Separating symVF, symVBP, and symVP relies on substantial work on homomorphism indistinguishability.

Cai, Fürer, & Immerman (1992); Dawar & Wang (2017); Roberson (2022); Neuen (2024); Dell, Grohe, & Rattan (2018); Dvořák (2010); Montacute & Shah (2024); Grohe (2020); Fluck, Spitzer, S. (2024); S. (2024); Dawar, Pago, S. (2026)

### Theorem (Single homomorphism polynomials)

*For every family of graphs $F_n$, the following holds*

$$(\mathrm{hom}_{F_n,n})_{n\in\mathbb{N}} \in \begin{cases} \mathsf{symVP} \\ \mathsf{symVBP} \\ \mathsf{symVF} \end{cases} \iff \begin{cases} \mathrm{tw}(F_n) \\ \mathrm{pw}(F_n) \\ \mathrm{td}(F_n) \end{cases} \in O(1).$$

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### Theorem

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**Question (Complexity Monotonicity)**

*When is the complexity of* $\sum \alpha_{F,n} \hom(F, \star)$ *governed by* $\max\{\mathrm{tw}(F) \mid \alpha_{F,n} \neq 0\}$*?*

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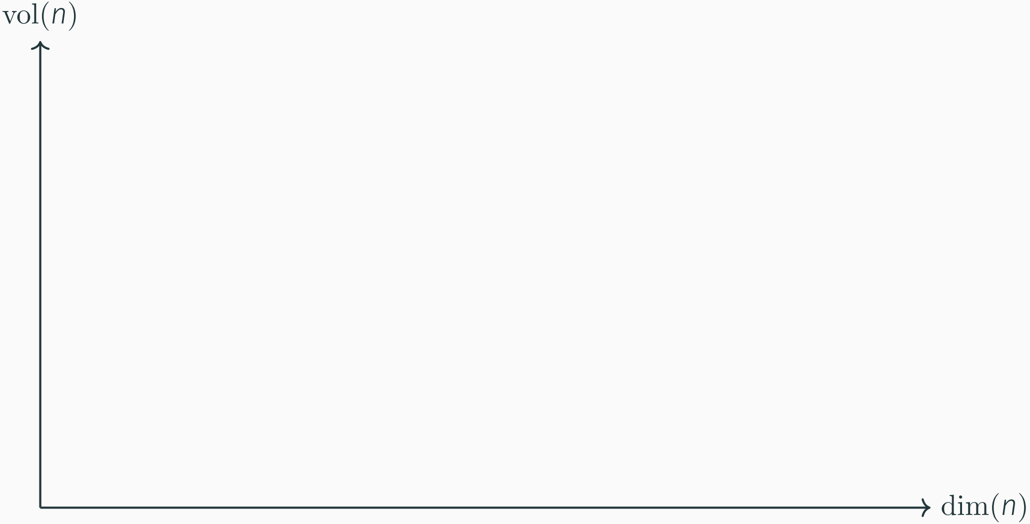
Observe that $\dim(n) \leq 2^{(\operatorname{vol}(n))^{O(1)}}$.




Graph motif parameters due to Curticapean, Dell, & Marx (2017). Parametrised Valiant's classes due to Bläser & Engels (2019).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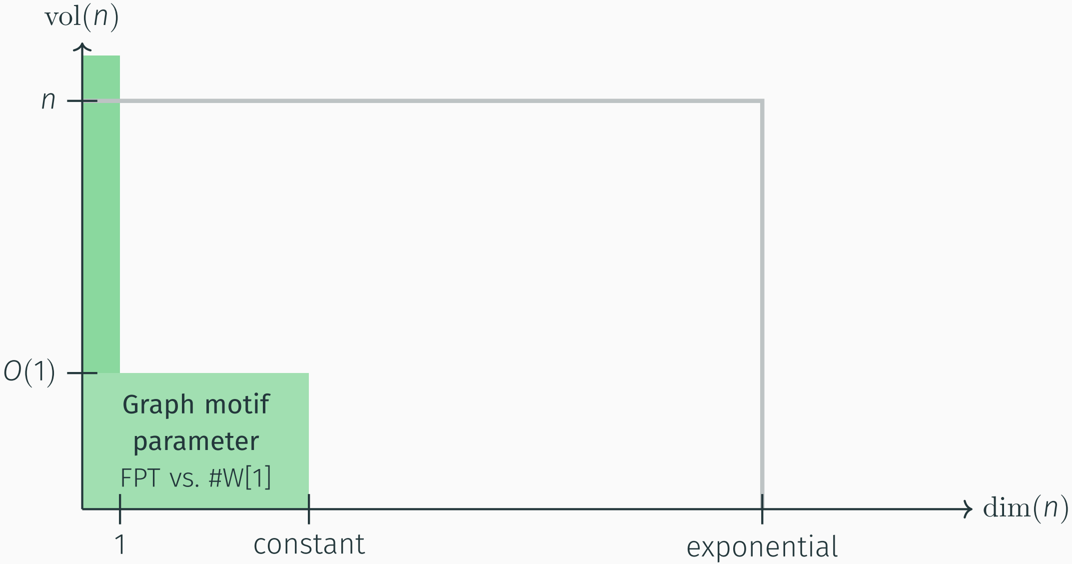


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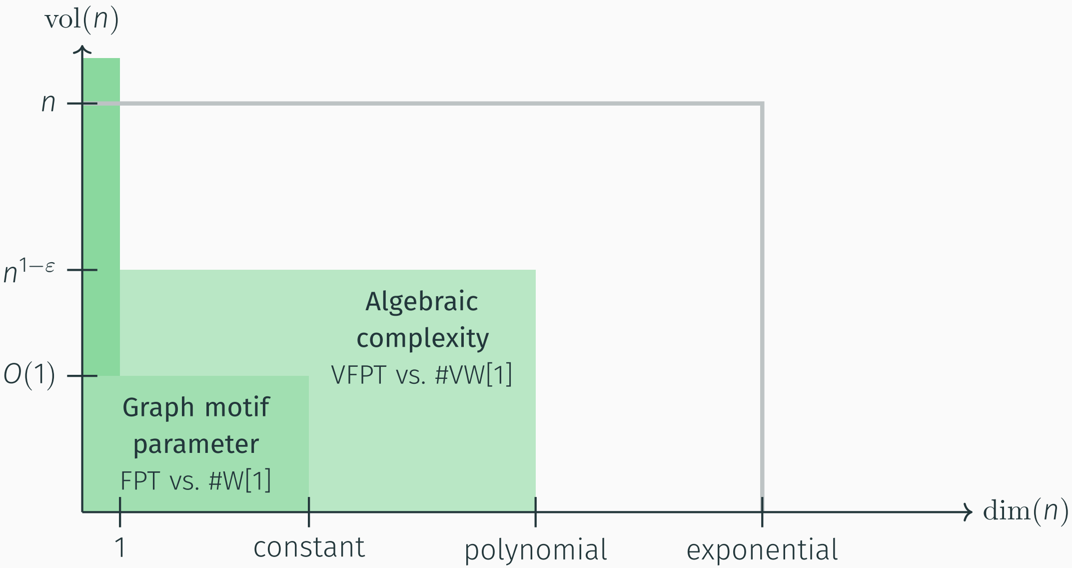


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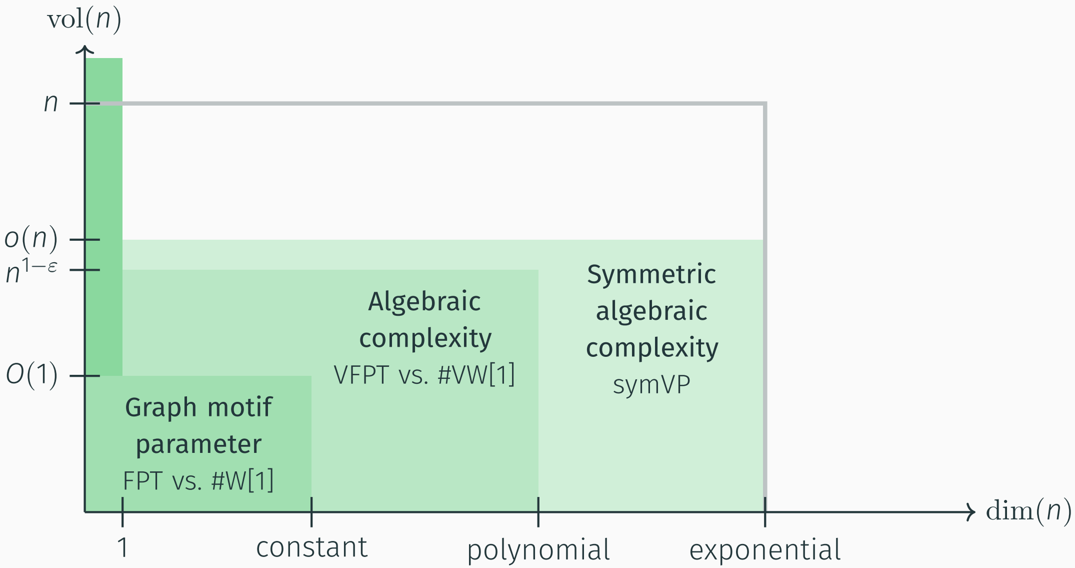


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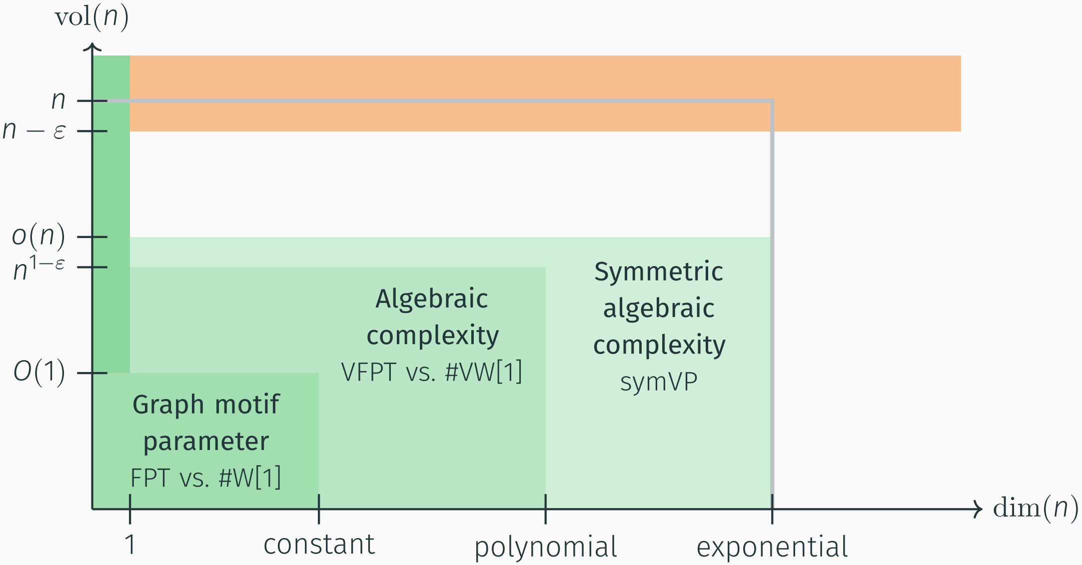


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- Next: linear combinations of linear volume or superpolynomial dimension.

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