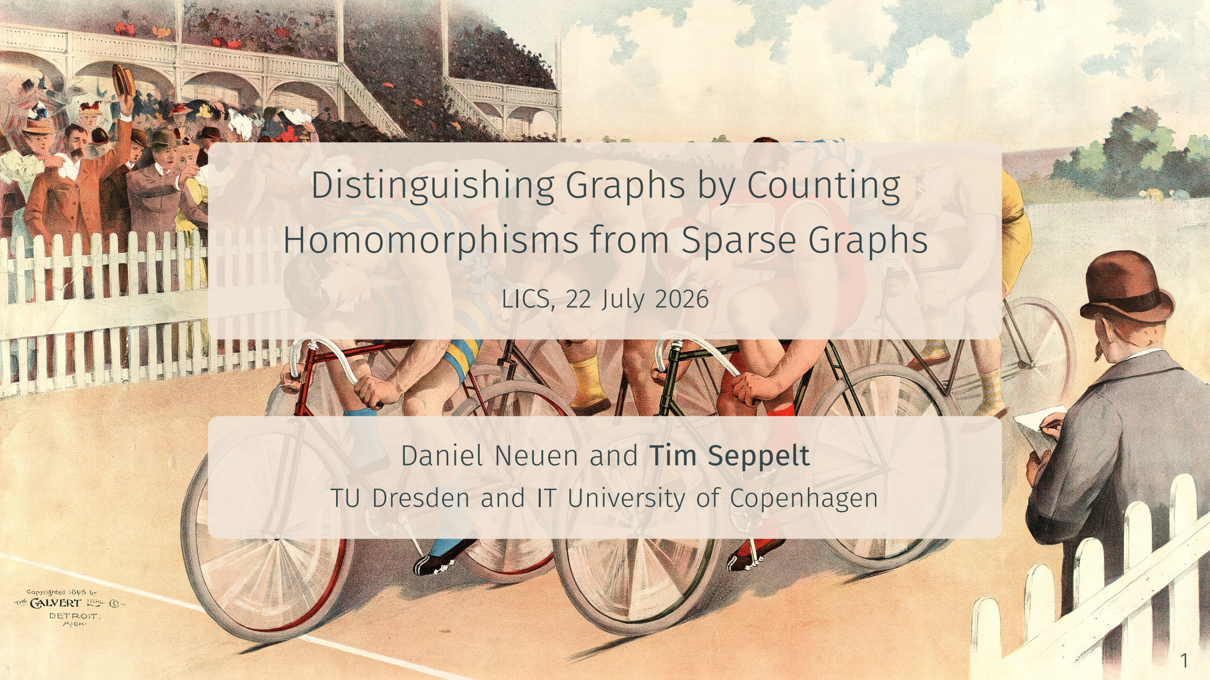
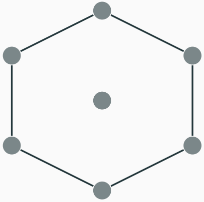
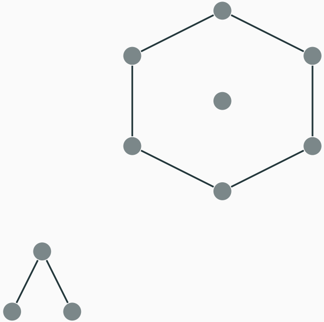


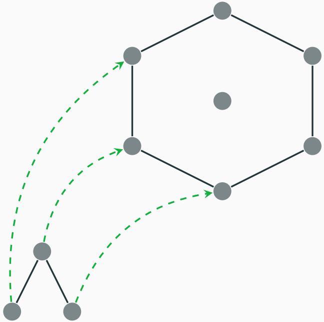
Distinguishing Graphs by Counting Homomorphisms from Sparse Graphs

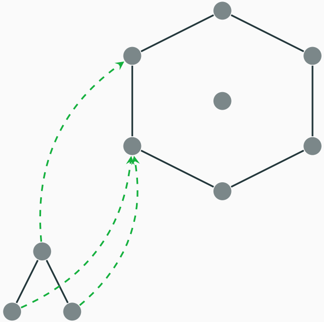
LICS, 22 July 2026

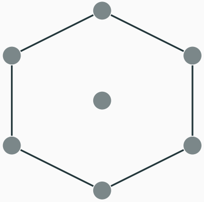
Daniel Neuen and **Tim Seppelt**
TU Dresden and IT University of Copenhagen

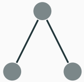












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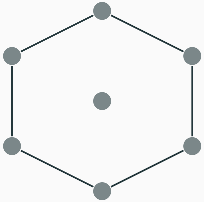

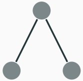

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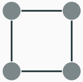

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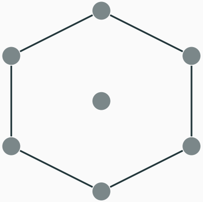
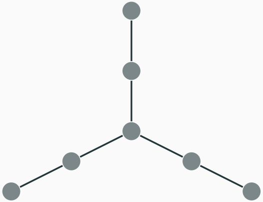
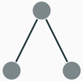
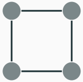
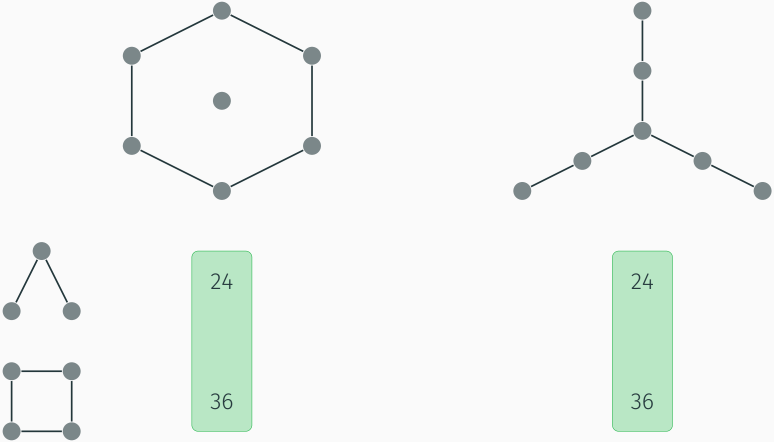
		
	24	24
	36	36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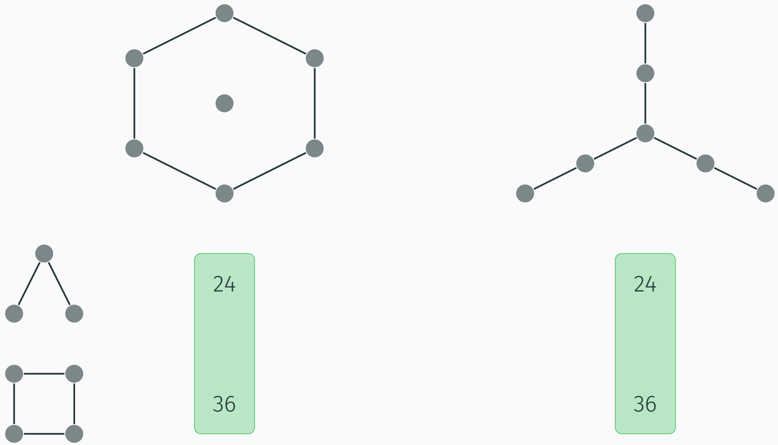
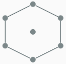
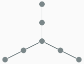


The graphs  and  are **homomorphism indistinguishable** over {, }.

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	Graph Neural Networks	Xu, Hu, Leskovec, & Jegelka (2018); Morris, Ritzert, Fey, Hamilton, Lenssen, Rattan, & Grohe (2019)

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Are all cospectral graphs isomorphic?

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Homomorphism indistinguishability over cycles is not isomorphism.

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Homomorphism indistinguishability over graphs of treewidth $< k$ is not isomorphism.

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Conjecture (Roberson (2022))

For every graph class $\mathcal{F}$ excluding some minor, the relation $\equiv_{\mathcal{F}}$ is not isomorphism.

forests

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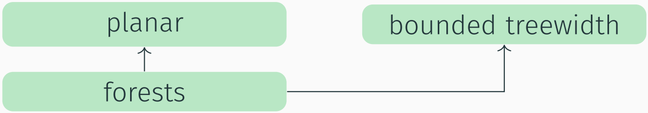

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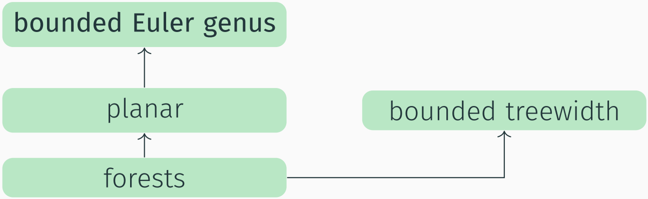

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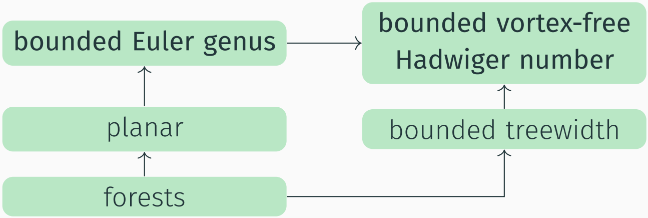

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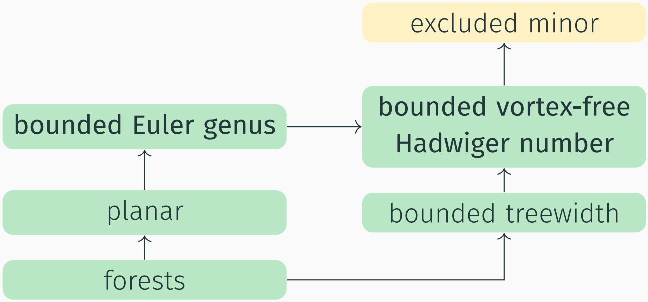

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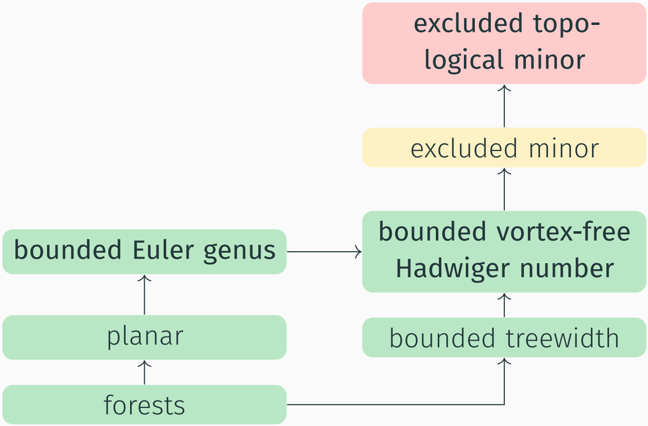

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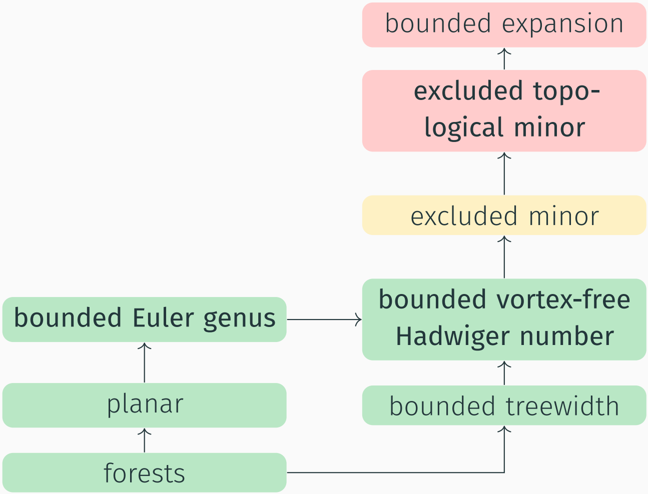

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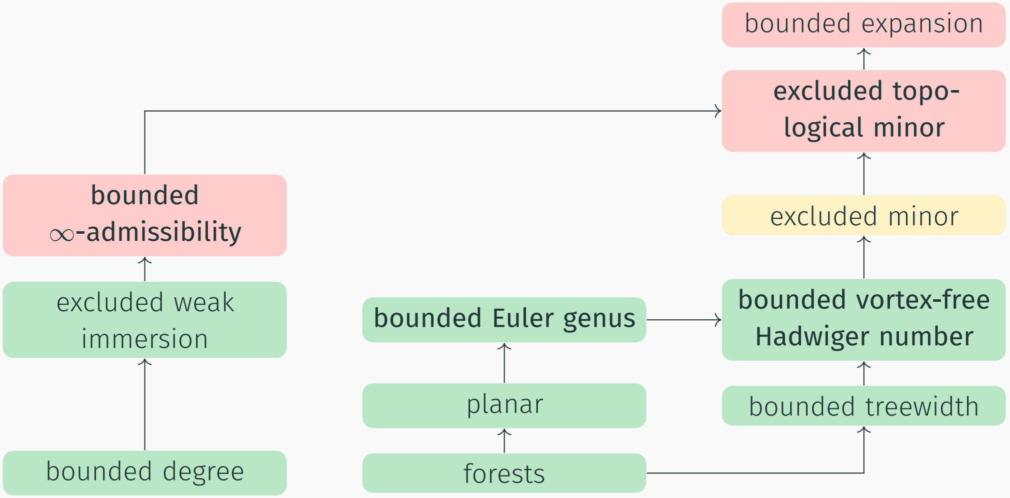

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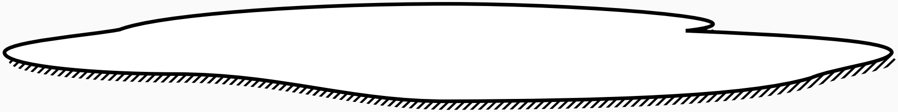
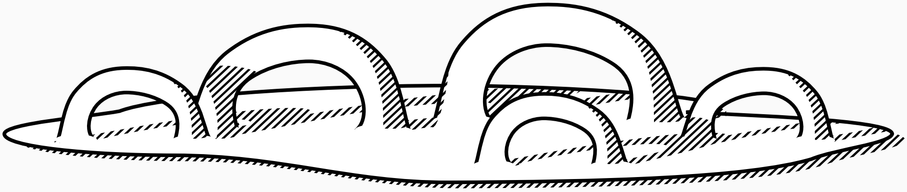


Figure by Felix Reidl.



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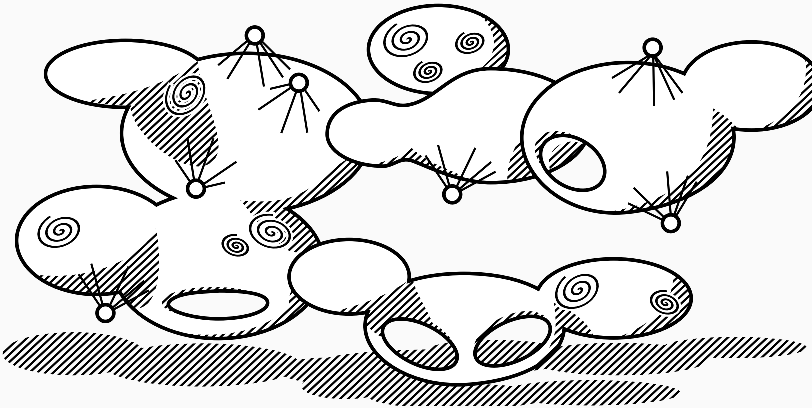

Figure by Felix Reidl. The *vortex-free Hadwiger number* was introduced by Thilikos & Wiederrecht (2022).

Techniques

Bounding Distinguishing Power without Characterizations

Given a class $\mathcal{F}$, we need to construct non-isomorphic G and H such that $G \equiv_{\mathcal{F}} H$.

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  - planar graphs: quantum isomorphism game — Atserias, Mančinska, Roberson, Šámal, Severini, & Varvitsiotis (2019)

Goals

- Show that $\equiv_{\mathcal{F}}$ is not isomorphism without characterizing $\equiv_{\mathcal{F}}$.

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Build on framework of Roberson (2022) for analyzing CFI graphs combinatorially.

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For a graph F and a connected graph G,

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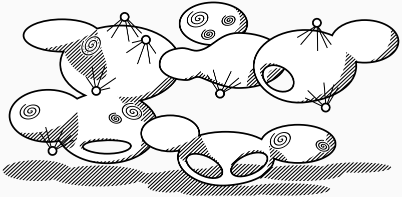
- Study how weak oddomorphisms interact with graph operations.

Theorem

For every $k \geq 0$, there exist non-isomorphic G and H that are homomorphism indistinguishable over all graphs of vortex-free Hadwiger number $\leq k$.



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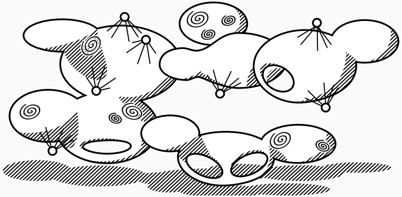
Black box ingredient: Planar graphs are w.o.-closed. Roberson (2022)



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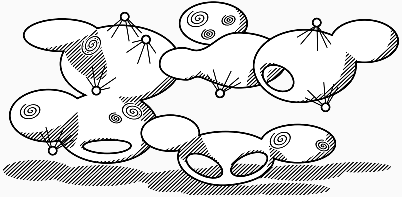

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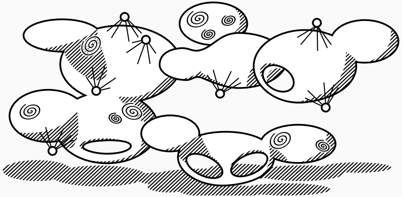

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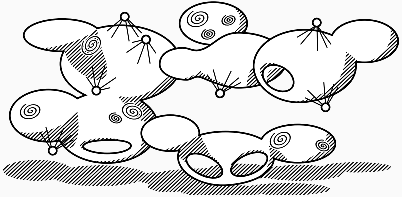

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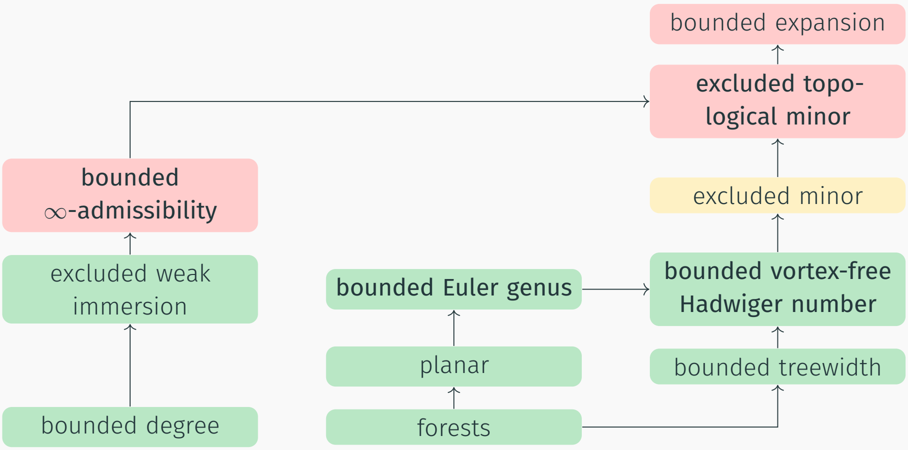
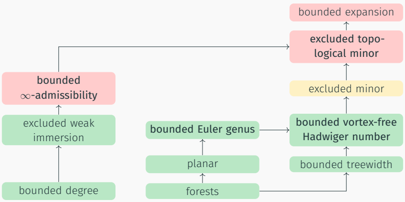
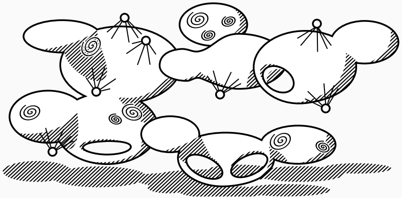


Future Work

- Combinatorial explanation for planar black box.





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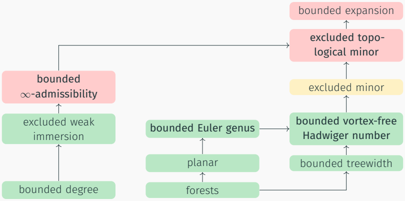
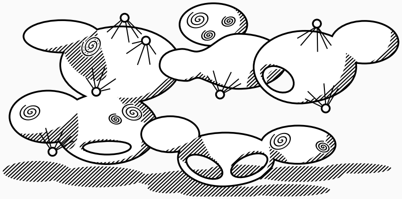
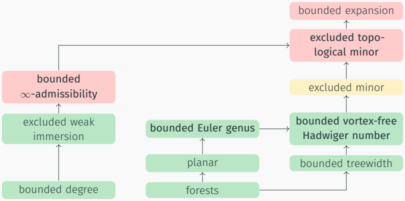
- Combinatorial explanation for planar black box.
- Roberson's (weak) conjecture

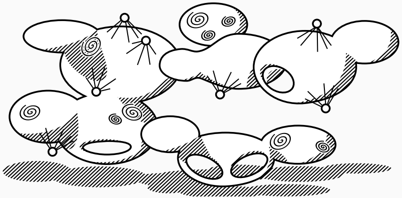




Future Work

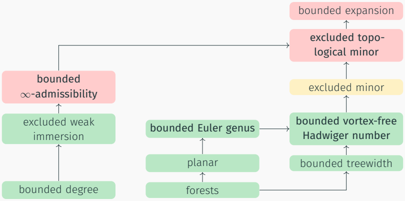
- Combinatorial explanation for planar black box.
- Roberson’s (weak) conjecture
- Roberson’s strong conjecture

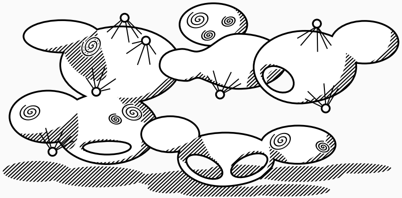




Future Work

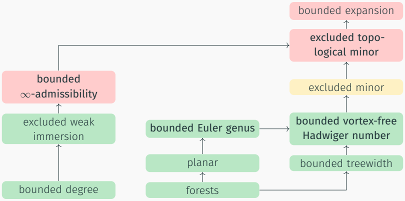
- Combinatorial explanation for planar black box.
- Roberson's (weak) conjecture
- Roberson's strong conjecture
- Dense graph classes.

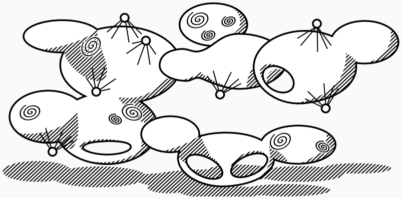




Future Work

- Combinatorial explanation for planar black box.
- Roberson's (weak) conjecture
- Roberson's strong conjecture
- Dense graph classes.
- Complexity of $\equiv_{\mathcal{F}}$ for $\mathcal{F} = \{\text{bounded genus}\}$ etc.





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